STATISTICAL ORBIT DETERMINATION ON THE BASIS OF UNEVEN DISTRIBUTED OBSERVATION DATA

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Abstract: *In the paper, a statistical approach, for satellite orbit determination, is presented. The developed method is assessed as a filtering procedure of the random noise that was come across like an additive to the measured parameter. The method aims at statistical estimation determination of parameter value, which had been previously considered. In addition, the algorithm allows the usage of various tracking stations. Data obtained could be referred to various moments of time. In that manner, the algorithm suggested may be used in complicated situations during satellite tracking process, such as getting loose the GPS scope. Because the algorithm is non–iterative, it is likely to be used in real time. Depending on the initial conditions, a various families of orbits could be also obtained. Conclusively, the numerical results obtained are collated with real satellite navigation data.*

Introduction

About an artificial satellite, whose both governing equations describing its motion and relevant initial conditions are previously known, its attitude in space could be determined by means of governing equations working out. In real, the amount of error of initial conditions, the mathematical model imperfection and the physical parameters are the cause of modeled trajectory deviation of the real one. Apparently, observations carried out always contain stochastic error. In order to estimate a state, which is closer to the real trajectory, the satellite observations carried out either from Earth tracking station or from another artificial satellite (for instance GPS), should be used with governing equations mentioned above altogether. Such an approach is used in Kalman filter procedure, which is used in the present report.

Preliminaries

According to what is shown in [1], equations describing satellite motion are as follows:

(1) $$\dot{\vec{X}} = F\left(\vec{X}, t\right),$$

where

(2) $$\vec{X} = \left\| X \quad Y \quad Z \quad \dot{X} \quad \dot{Y} \quad \dot{Z} \quad \beta \right\|^{T}.$$

In the formula above, “X” denotes state vector, consisting of position and velocity vector components and m – dimensional vector of constant parameters as well. In this way vector “X” dimension is N = 6 + M.
Equation (1) can be linearized, in accordance with power series law, in relation to the state vector, computed along the reference trajectory “X*”:

(3) $$\dot{\vec{X}} = \dot{\vec{X}}\left(t\right) + \left[\frac{\partial \dot{\vec{X}}\left(t\right)}{\partial \vec{X}\left(t\right)}\right]^{*} \left[\vec{X}\left(t\right) - \vec{X}^{*}\left(t\right)\right] + higher \ \ order \ \ terms \ .$$

Index “*” denotes that the value is computed about the reference trajectory. If terms of order higher than first are neglected and the following relation is used as well:

(4) $$\vec{x}\left(t\right) = \dot{\vec{X}}\left(t\right) - \vec{X}^{*}\left(t\right).$$

Power series (3) yields

(5) $$\dot{\vec{x}}\left(t\right) = \left[\frac{\partial \dot{\vec{X}}\left(t\right)}{\partial \vec{X}\left(t\right)}\right]^{*} \vec{x}\left(t\right).$$

Also, if the following variable is defined

(6) $$A(t)=\left[\frac{\partial \dot{\vec{X}}(t)}{\partial \vec{X}(t)}\right]^*$$

then the following expression is obtained:

(7) $$\dot{\vec{x}}(t)=A(t)\vec{x}(t).$$

Equality (7) represents a system first order Ordinary Differential Equations. The matrix A(t) has dimension NxN and is computed about the reference trajectory X*(t), previously given. Regarding that dβ/dt=0, it yields:

(8) $$\frac{\partial \dot{\beta}}{\partial \vec{X}(t)}=0\ .$$

Since equation (5) is linear, the solution could be written in following manner:

(9) $$\vec{x}(t)=\frac{\partial \vec{x}(t)}{\partial \vec{x}(0)}\vec{x}(0)=\frac{\partial \vec{X}(t)}{\partial \vec{X}(0)}\vec{x}(0).$$

The last quantity follows after

(10) $$\frac{\partial \vec{x}(t)}{\partial \vec{x}(0)}=\frac{\partial\left[\vec{X}(t)-\vec{X}^*(t)\right]}{\partial\left[\vec{X}(0)-\vec{X}^*(0)\right]}=\frac{\partial \vec{X}(t)}{\partial \vec{X}(0)}.$$

There are conditions, according to equation (9), which satisfies equation (5). If the following variable is defined

(11) $$\Phi(t_0,t)=\frac{\partial \vec{X}(t)}{\partial \vec{X}(0)},$$

equation (9) takes the form

(12) $$\vec{x}(t)=\Phi(t_0,t)\vec{x}(0).$$

If the last equation is differentiated in relation to the time, it follows:

(13) $$\dot{\vec{x}}(t)=\dot{\Phi}(t_0,t)\vec{x}(0).$$

Because of equations (5) and (13) equalization, it could be written:

(14) $$\left[\frac{\partial \dot{\vec{X}}(t)}{\partial \vec{X}(t)}\right]^*\vec{x}(t)=\dot{\Phi}(t_0,t)\vec{x}(0).$$

In case of substitution of “x(t)”, with the relevant value obtained by (12), it follows

(15) $$\left[\frac{\partial \dot{\vec{X}}(t)}{\partial \vec{X}(t)}\right]^*\Phi(t_0,t)\vec{x}(0)=\dot{\Phi}(t_0,t)\vec{x}(0).$$

In case of equalization of the relevant coefficients in front of “x(0)” further, the following Ordinary Differential Equation is obtained, in relation to Φ(t0,t):

(16) $$\dot{\Phi}(t_0,t)=A.\Phi(t_0,t).$$

The initial conditions are:

(17) $$\Phi(t_0,t_0)=I\ .$$

Matrix Φ(t0,t) is called *State Transition Matrix*. It features with the properties: Φ(tk,tk) = I, Φ(ti,tk) = Φ(ti,tj) * Φ(tj,tk), Φ(ti,tk) = inv(Φ(tk,ti). Each time when equations (16) and (17) are satisfied, the solution of the differential equation

(18) $$\dot{\vec{x}}(t)=A\vec{x}(t)$$

is given by means of solving the algebraic equation:

(19) $$\vec{x}(t)=\Phi(t_0,t)\vec{x}(0)$$

The matrix Φ(t0,t) can be obtained as follows. Let

(20) $$\Phi(t_0,t)\equiv\frac{\partial \vec{X}(t)}{\partial \vec{X}(0)}\equiv\left\|\begin{matrix}\phi_1(t_0,t)\\ \phi_2(t_0,t)\\ \phi_3(t_0,t)\end{matrix}\right\|=\left\|\begin{matrix}\frac{\partial \vec{r}(t)}{\partial \vec{X}(0)} & \frac{\partial \dot{\vec{r}}(t)}{\partial \vec{X}(0)} & \frac{\partial \beta(t)}{\partial \vec{X}(0)}\end{matrix}\right\|^T.$$

The matrix φ3(t0,t) has dimension MxN and has constant elements. It could be conditionally divided into zero matrix Mx6 and identity matrix MxM. The number “M” has the same dimension of vector “X”. Because of this, the upper half 6xN part of equation (16) is only necessary to work out.
Equation (16) could be written as 2nd order Ordinary differential Equation. It could be shown, if differentiate the equation (20) in accordance with time:

(21)
$$\dot{\Phi}(t_0,t) \equiv \frac{\partial \dot{\vec{X}}(t)}{\partial \vec{X}(0)} \equiv \begin{Vmatrix} \dot{\phi}_1(t_0,t) \\ \dot{\phi}_2(t_0,t) \\ \dot{\phi}_3(t_0,t) \end{Vmatrix} = \begin{Vmatrix} \frac{\partial \dot{\vec{r}}(t)}{\partial \vec{X}(0)} & \frac{\partial \ddot{\vec{r}}(t)}{\partial \vec{X}(0)} & 0 \end{Vmatrix}^T = .$$
$$= \begin{Vmatrix} \frac{\partial \dot{\vec{r}}(t)}{\partial \vec{X}(0)} & \frac{\partial \ddot{\vec{r}}(t)}{\partial \vec{X}(0)} & 0 \end{Vmatrix}^T_{nxn} \begin{Vmatrix} \frac{\partial \vec{X}(t)}{\partial \vec{X}(0)} \end{Vmatrix}_{nxn}$$

In the last equation, the symbol “0” denotes zero matrix MxN. Regarding first row of equation (21), it follows that

(22) $$\ddot{\phi}_1 = \frac{\partial \ddot{\vec{r}}(t)}{\partial \vec{X}(0)} = \dot{\phi}_2 .$$

Therefore, this system of second order Ordinary Differential Equations can be solved according to Φ(t0,t):

(23)
$$\ddot{\phi}_1(t_0,t) = \frac{\partial \ddot{\vec{r}}(t)}{\partial \vec{X}(0)} = \frac{\partial \ddot{\vec{r}}(t)}{\partial \vec{X}(t)} \frac{\partial \vec{X}(t)}{\partial \vec{X}(0)} =$$
$$= \begin{Vmatrix} \frac{\partial \ddot{\vec{r}}(t)}{\partial \vec{r}(t)} & \frac{\partial \ddot{\vec{r}}(t)}{\partial \dot{\vec{r}}(0)} & \frac{\partial \ddot{\vec{r}}(t)}{\partial \beta(t)} \end{Vmatrix}_{3xn} \begin{Vmatrix} \frac{\partial \vec{r}(t)}{\partial \vec{X}(0)} & \frac{\partial \dot{\vec{r}}(t)}{\partial \vec{X}(0)} & \frac{\partial \beta(t)}{\partial \vec{X}(0)} \end{Vmatrix}^T_{nxn}$$

or simply

(24) $$\ddot{\phi}_1(t_0,t) = \frac{\partial \ddot{\vec{r}}(t)}{\partial \vec{r}(t)} \phi_1(t_0,t) + \frac{\partial \ddot{\vec{r}}(t)}{\partial \dot{\vec{r}}(t)} \dot{\phi}_2(t_0,t) + \frac{\partial \ddot{\vec{r}}(t)}{\partial \beta(t)} \phi_3(t_0,t) .$$

Using initial conditions

(25)
$$\phi_1(t_0,t_0) = \begin{Vmatrix} \|I\|_{3x3} \|0\|_{3x(n-3)} \end{Vmatrix}$$
$$\dot{\phi}_1(t_0,t_0) = \dot{\phi}_2(t_0,t_0) = \begin{Vmatrix} \|0\|_{3x3} \|I\|_{3x3} \|0\|_{3xm} \end{Vmatrix} ,$$

equation (24) could be solved as system 2nd order Ordinary Differential Equations 3xN, instead of system first order Ordinary Differential Equations 6xN (see eq. (21)).. Partial derivatives here are computed about the reference state. The solution then, about system MxN and initial conditions dφ3(t0,t)/dt = 0, is trivial:

(26) $$\phi_3(t_0,t) = \begin{Vmatrix} \|0\|_{mx6} \|I\|_{mxm} \end{Vmatrix} .$$

During computations, equation (24) could be written as a system first order Ordinary Differential Equations NxN:

(27) $$\begin{Vmatrix} \dot{\phi}_1(t_0,t) \\ \dot{\phi}_2(t_0,t) \\ \dot{\phi}_3(t_0,t) \end{Vmatrix} = \begin{Vmatrix} \|0\|_{3x3} & \|I\|_{3x3} & \|0\|_{3xm} \\ \begin{Vmatrix} \frac{\partial \ddot{\vec{r}}(t)}{\partial \vec{r}(t)} \end{Vmatrix}_{3x3} & \begin{Vmatrix} \frac{\partial \ddot{\vec{r}}(t)}{\partial \dot{\vec{r}}(t)} \end{Vmatrix}_{3x3} & \begin{Vmatrix} \frac{\partial \ddot{\vec{r}}(t)}{\partial \beta(t)} \end{Vmatrix}_{3xm} \\ \|0\|_{mx3} & \|0\|_{mx3} & \|0\|_{mxm} \end{Vmatrix}^*_{mxn} \begin{Vmatrix} \phi_1(t_0,t) \\ \phi_2(t_0,t) \\ \phi_3(t_0,t) \end{Vmatrix}_{nxn} .$$

Also, it yields

(28) $$\dot{\Phi}(t_0,t) = A(t).\Phi(t_0,t).$$

If consider that the satellite observations are done about moments of time t1, t2,…, tl, then the functional form of the observation equations could be expressed as follows:

(29) $$\vec{Y}_i = G(\vec{r}_i, \vec{r}_{si}, t_i) + \vec{\varepsilon}_i , i = 1,2,..., I ,$$

where “r” and “rs” are geocentric positions of the space ship and the tracking station about time “ti”. Symbol “ε” denotes measured error which is admitted presenting the white Gauss noise. After substitution about the state vector, equation (29) takes the form:

(30) $$\vec{Y}_i = G(\vec{X}_i, t_i) + \vec{\varepsilon}_i , i = 1,2,..., I .$$

In that manner, the real observation is considered to be a nonlinear function of “analytical” observation G(X,t) and measured random noise ε.
In order both vectors of observation and state to be related to each other in linear manner, a method from linear estimation theory is applied. If both the reference X* and the real X trajectory remain close enough, during time interval regarded, then a linear dependence between observation deviation y = Y – Y* and the state deviation x = X – X* could be established. In linear problem of orbit determination, the estimation being asked refers to a deviation from the reference trajectory, which is given in advance.
Both vectors “x” and “y” could be obtained by means of solving the following system of Ordinary Differential Equations with variable coefficients (see eq. (18)):

$$\begin{aligned} \dot{\vec{x}} &= A(t)\vec{x} \\ \vec{y}_i &= \tilde{H}_i \vec{x}_i + \vec{\varepsilon}_i \end{aligned} , \tag{31}$$

where

$$A(t) = \frac{\partial \dot{\vec{X}}\left(\vec{X}^*, t\right)}{\partial \vec{X}}, \tilde{H} = \frac{\partial G\left(\vec{X}^*, t\right)}{\partial \vec{X}} . \tag{32}$$

The solution of the first equation in system (31) is obtained as a solution of equation (19), while the solution of the second equation could be written as functions of state variable, as follows:

$$\begin{aligned} \vec{y}_1 &= \tilde{H}_1 \Phi(t_0, t_1)\vec{x}_1 + \vec{\varepsilon}_1 \\ \vec{y}_2 &= \tilde{H}_2 \Phi(t_1, t_2)\vec{x}_2 + \vec{\varepsilon}_2 \\ &\ldots \\ \vec{y}_i &= \tilde{H}_i \Phi(t_{i-1}, t_i)\vec{x}_i + \vec{\varepsilon}_i \end{aligned} \tag{33}$$

Equation above contains M observations (“y” is vector Mx1) and N unknown components of the state (“x” has dimension Nx1). Apparently, “ε” is vector Mx1, “H” is linear operator MxN. If M>N then the solution , in accordance with the Least Squares Method, could be obtained as follows:

$$\hat{\vec{x}}(0) = \left(\tilde{H}^T \tilde{H}\right)^{-1} \tilde{H}^T \vec{y} . \tag{34}$$

More details, about Linear Estimation theory applied to the Statistical Orbit Determination problem, are given in paper [2]. More information about the Statistical Orbit Determination about satellite constellation is given in either thesis [3] or paper [4].

Results

In the present paper, a case of satellite motion, under the influence of central force gravitational field, is considered. The reference frame used is a spherical rotational. According to Newton second law, about the satellite generalized coordinate, it could be written that:

$$\ddot{\vec{r}}(t) = -\frac{\mu \vec{r}(t)}{\left|\vec{r}(t)\right|^3} . \tag{35}$$

By means of program package “Mathematica”® usage, written by Wolfram Research, Inc., the following expressions were obtained:

- state vector

$$\vec{X}(t) = \left\| X \quad Y \quad Z \quad \dot{X} \quad \dot{Y} \quad \dot{Z} \quad \mu \quad X_s \quad Y_s \quad Z_s \right\|^T ; \tag{36}$$

- state vector derivative

$$\dot{\vec{X}}(t) = F\left(\vec{X}, t\right) = \left\| \dot{X} \quad \dot{Y} \quad \dot{Z} \quad -\frac{\mu X}{|\vec{r}|^3} \quad -\frac{\mu Y}{|\vec{r}|^3} \quad -\frac{\mu Z}{|\vec{r}|^3} \quad 0 \quad 0 \quad 0 \quad 0 \right\|^T ; \tag{37}$$

- coefficient matrix

$$A(t)=\left[\frac{\partial\dot{\vec{X}}(t)}{\partial\vec{X}}\right]^{*}=$$

(38)
$$=\left\|\begin{matrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ -\frac{\mu}{|\vec{r}|^3}+\frac{3\mu X^2}{|\vec{r}|^5} & \frac{3\mu XY}{|\vec{r}|^5} & \frac{3\mu XZ}{|\vec{r}|^5} & 0 & 0 & 0 & -\frac{X}{|\vec{r}|^3} & 0 & 0 & 0\\ \frac{3\mu YX}{|\vec{r}|^5} & -\frac{\mu}{|\vec{r}|^3}+\frac{3\mu Y^2}{|\vec{r}|^5} & \frac{3\mu YZ}{|\vec{r}|^5} & 0 & 0 & 0 & -\frac{Y}{|\vec{r}|^3} & 0 & 0 & 0\\ \frac{3\mu ZX}{|\vec{r}|^5} & \frac{3\mu ZY}{|\vec{r}|^5} & -\frac{\mu}{|\vec{r}|^3}+\frac{3\mu Z^2}{|\vec{r}|^5} & 0 & 0 & 0 & -\frac{Z}{|\vec{r}|^3} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}\right\|^{*}$$

- observation vector, also elements of vector G(X,t)

$$\vec{Y}(t)=\|\rho \quad \dot{\rho}\|^{T}$$

(39) $$|\vec{\rho}|=\sqrt{(X-X_s)^2+(Y-Y_s)^2+(Z-Z_s)^2}\ .$$

$$\frac{\partial\vec{\rho}}{\partial t}=\frac{\dot{X}(X-X_s)+\dot{Y}(Y-Y_s)+\dot{Z}(Z-Z_s)}{|\vec{\rho}|}$$

- observation matrix

(40) $$\tilde{H}=\left\|\begin{matrix} \frac{\partial|\vec{\rho}|}{\partial X} & \frac{\partial|\vec{\rho}|}{\partial Y} & \frac{\partial|\vec{\rho}|}{\partial Z} & \frac{\partial|\vec{\rho}|}{\partial\dot{X}} & \frac{\partial|\vec{\rho}|}{\partial\dot{Y}} & \frac{\partial|\vec{\rho}|}{\partial\dot{Z}} & \frac{\partial|\vec{\rho}|}{\partial\mu} & \frac{\partial|\vec{\rho}|}{\partial X_s} & \frac{\partial|\vec{\rho}|}{\partial Y_s} & \frac{\partial|\vec{\rho}|}{\partial Z_s}\\ \frac{\partial|\dot{\vec{\rho}}|}{\partial X} & \frac{\partial|\dot{\vec{\rho}}|}{\partial Y} & \frac{\partial|\dot{\vec{\rho}}|}{\partial Z} & \frac{\partial|\dot{\vec{\rho}}|}{\partial\dot{X}} & \frac{\partial|\dot{\vec{\rho}}|}{\partial\dot{Y}} & \frac{\partial|\dot{\vec{\rho}}|}{\partial\dot{Z}} & \frac{\partial|\dot{\vec{\rho}}|}{\partial\mu} & \frac{\partial|\dot{\vec{\rho}}|}{\partial X_s} & \frac{\partial|\dot{\vec{\rho}}|}{\partial Y_s} & \frac{\partial|\dot{\vec{\rho}}|}{\partial Z_s} \end{matrix}\right\|^{*}$$

Because of complexity of expressions obtained, regarding last formulae, it is not possible to be published here.
The algorithm of the observed data processing, in accordance with the theory established so far, is shown in the figure below. In addition, the results obtained, which are compared with the measured data, are shown in the next figure. It is assumed that data presented are distributed in uneven manner.

Conclusion

In the present report, a method for satellite's orbit kinematic parameters determination, on the basis of uneven distributed observations, were presented. The main feature of the method is linearization of governing system equations. Therefore, the solution of these equations was found in relation to a small deviation from the reference trajectory, which is previously given.
During the computational process, a Sequential Least squares Method has been used. The choice between either Batch or Sequential filter depends on the current problem being solved. Also, it makes no difference in efforts which was put into when carrying out a research, using one of these methods.

References:

1. T a p l e y B. D., B. E. S c h u t z, G. H. B o r n. Statistical Orbit Determination, Elsevier Academic Press, 2004
2. T a p l e y B. D. Statistical Orbit Determination Theory, in Tapley, B. D., V. Szebehely, eds., recent Advances in Dynamical Astronomy, D. Reidel, hingham, Massachussets, 1973
3. R i m H., TOPEX Orbit Determination using GPS Tracking system, PhD thesis, The University of Texas at Austin, 1992
4. T a p l e y B. D., J. C. R i e s. Precision Orbit Determination for Earth Observation Systems, in encyclopedia of Space Science and Technology, pp. 341 – 355, Whylie and Sons, Hoboken, New Jersey, 2003.