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Qualitative analysis of the free processes in a generalized linear oscillating circuit with periodic parameters. I. Structure of the differential equations and classification of the free processes in Hamiltonian oscillating circuits[1]

*Nikolai Birjuk, Vladimir Damgov**

Voronej State University, Department of Physics
394693 Voronej, Russia
**Space Research Institute, Bulgarian Academy of Sciences*

Introduction

The analysis of linear oscillating systems with variable parameters, and particularly of such with parameters that remain periodical in time, is of fundamental importance for the investigation of oscillating systems in a general form [1]. The parameters of nonlinear oscillating systems depend on the voltage applied and on the currents that flow through them, and these, in their turn, are functions of time. Thus, in the long run, nonlinear systems are also systems with variable parameters. In this connection a principle of linear linkage is formulated in mathematics [2]. It is related to the idea that the phenomena and properties of nonlinear systems can be realized (in the sense of simulated) for each specific (particular) case in the respective linear systems with variable parameters.

Qualitative analysis assumes considerable importanse in the investiga-

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tion of complex oscillating systems, since it allows of identifying the most general features of system behaviour.

The paper reveals a general method for analyzing linear systems with periodic and almost periodic parameters.

The generalized linear oscillating circuit (Fig. 1) with periodic parameters is called so forth to play an important part in the theory of nonlinear oscillations and in radiophysics. Its direct significance is that it is used in radiophysics as a signal oscillating circuit parametric amplifier or generator, in the process of oscillation modulation in generator systems, etc. Its indirect significance is conditioned by the fact that it is a heuristic model of nonlinear autonomous and nonautonomous second-order systems employed in the analysis of process stability in such systems.

The paper provides an analysis of the free processes in an oscillating circuit from the point of view of their bounded or unbounded nature, or, in other words, it studies issues of the stability and instability of the oscillating circuit. The difficulty of the task is predeterminated by its most general set-up, which requires the application of complex mathematical techniques. Yet such a general perspective of the task makes it interesting from a practical point of view, since the parametric oscillating circuit is quite rich in particular cases, but principle no specific particular case can reveal the overall diversity of possibilites for the oscillating circuit.

Further down we have quoted a basic system of two linear differential equations of the generalized parametric oscillating circuit, as well as some particular cases derived from the basic system through variable substitution. The attention is mostly focused on the canonical second-order system, to which the basic system of circuit equations is reduced.

Mathieu's equation is equivalent to a rather particular case of a canonical system with periodic coefficients. It is well-known that Mathieu's various equations can be classified within a definite set of classes by zones of stability and instability. These zones can be presented in a two-dimensional plane as areas with sufficiently complex form which intertwine and overlap in a complex way. It turns out that the canonical systems of a general type have analogous properties but the respective stability and instability areas are fixed in cylindrical space obtained by rotating the plane round an axis lying in this plane. The respective results are obtained by employing incomparably more complex methods than in the case of Mathieu's equation and a broader system of mathematical concepts.

Structure of the differential equations describing linear oscillating systems with positive parameters

Linear oscillating circuits can be described by applying a first-order vector linear differential equation

$$\frac{d}{dt}z = A(t)z + f(t), \tag{1}$$

where z is a n-dimensional vector, whose elements can represent capacitor charges, magnetic flux running through inductances, etc. $A(t)$ is a $n \times n$ -dimensional matrix whose elements can be expressed by circuit parameters (inductances, capacitances, resistances), $f(t)$ is a free n-dimensional vector, whose components are determined by the electromotive forces connected to the circuit and by the parameters of the system.

In order to identify the structure of equation (1), we shall initially analyze the following equation of the free processes with "frozen" (time-independent) parameters:

$$\frac{d}{dt}x = Ax, \tag{2}$$

where A = const. If all the parameters of the system are positive, the solution will satisfy the condition: $\lim\limits_{t\to\infty} x(t) = 0$.

It is obvious that the latter equation meets the condition that SpA<0, (SpA is the sum total of the main diagonal terms of matrix A).

L e m m a 1. Any radiophysical system with constant positive parameters, containing active resistances with currents flowing through them, is described by a system of differential equations with constant coefficients, whose matrix includes non-positive main diagonal elements, at least one of which is negative.

L e m m a 2. The main diagonal in the matrix of the first-order vector differential equation of a radio circuit with constant positive parameters, containing only real reactances (with losses), consists only of negative elements.

The condition that SpA<0 and the following Lemma are valid for circuits made up of ideal reactances with constant positive parameters.

L e m m a 3. The matrix of the vector differential equation of a radio circuit containing only ideal reactances with constant positive parameters has a zero main diagonal.

The Lemmas formulated above are also valid for a certain subclass of linear radio circuits (we shall term it a structurally stationary one) with variable positive parameters. They are characterized by the fact that matrix A in (2) contains no derivatives of the circuit parameters with respect to the time. The following theorem can be formulated in this connection.

Theorem 1. Any radio circuit with variable parameters containing no capacitive loops or inductive nodes are structurally stationary.

The proof of Theorem 1 is based on Kirchhoff's laws. It demonstrates that the derivatives of the circuit parameters with respect to time emerge only after excluding one of the charges in a capacitive loop, or one of the magnetic fluxes in an inductive node.

With a view to extending the scope of action of the Lemmas on structurally stationary circuits considered above, the latter can be reformulated as separate theorems.

Two connected oscillating circuits with intrinsic capacitive coupling can serve as an example off a structurally stationary circuit. Two connected oscillating circuits with external capacitive coupling cannot be regarded as a structurally stationary circuit, since in this case the three capacitances form a capacitive loop.

Vector differential equation describing a linear oscillating circuit with time-dependent parameters

The free process in a linear generalized oscillating circuit with changing parameters (Fig. 1) is described by a linear vector differential equation

$$\frac{d}{dt}x = A(t)x, \tag{3}$$

$x = \text{colon}\,(x_1, x_2), x_1 = \dfrac{q}{q_{00}}$ - normalized charge of the capacitor , $x_2 = \dfrac{\Phi}{\Phi_{00}}$ - normalized magnetic flux of the inductance, $A(t) = \{a_{ij}(t)\}, \quad i, j = 1,2,$

$$a_{11}(t) = -t_{00}\frac{G(t)}{C(t)} < 0,\ a_{12}(t) = -\frac{t_{00}\Phi_{00}}{q_{00}L(\psi)} < 0,\ a_{21}(t) = \frac{t_{00}q_{00}}{\Phi_{00}C(t)} > 0,$$

$$a_{22}(t) = -\frac{t_{00}R(t)}{L(t)} < 0.$$

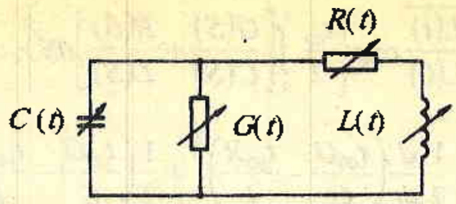

Fig. 1. Linear generalized oscillating circuit with changing parameters

If x_2 is excluded from system (3), the result will be a scalar differential equation with regard to x_1. A scalar equation with respect to x_2 can be obtained in an analogous way. These are equations of the type

(4) $$\frac{d^2x}{dt^2}+a_1(t)\frac{dx}{dt}+a_2(t)x=0,$$

where

- in the case of $x=x_1, a_1(t)=\frac{d}{dt}\ln L(t)+t_{00}\left[\frac{G(t)}{C(t)}+\frac{R(t)}{L(t)}\right]$,

$$a_2(t)=\frac{d}{dt}\left[\frac{t_{00}G(t)}{C(t)}\right]+\frac{t_{00}G(t)}{C(t)}\frac{d}{dt}\ln L(t)+\frac{t_{00}}{L(t)C(t)}\left[1+R(t)G(t)\right];$$

- in the case of $x=x_2$, respectively $a_1(t)=\frac{d}{dt}\ln C(t)+t_{00}\left[\frac{R(t)}{L(t)}+\frac{G(t)}{C(t)}\right]$,

$$a_2(t)=\frac{d}{dt}\left[\frac{t_{00}R(t)}{L(t)}\right]+\frac{t_{00}R(t)}{L(t)}\frac{d}{dt}\ln C(t)+\frac{t_{00}}{L(t)C(t)}\left[1+G(t)R(t)\right].$$

A substitution of variable in (4) by $x=\exp\left[-\frac{1}{2}\int_0^t a_1(S)dS\right]y$ yields

(5) $$\frac{d^2y}{dt^2}+P(t)y=0,\quad P(t)=a_2(t)-\frac{1}{4}a_1^2(t)-\frac{1}{2}\frac{da_1(t)}{dt}.$$

Given the assumptions that in (4) $x=x_1$, then

$$y = y_1 = x_1\sqrt{\frac{L(t)}{L(0)}}\exp\left\{\frac{t_{00}}{2}\int_0^t\left[\frac{G(S)}{C(S)}+\frac{R(S)}{L(S)}\right]dS\right\},$$

$$P(t) = \frac{t_{00}^2}{LC}(1+RG)+\frac{1}{2}\frac{d}{dt}\left(\frac{t_{00}G}{C}-\frac{t_{00}R}{L}\right)^2+\frac{1}{2}\left(\frac{t_{00}G}{C}-\frac{t_{00}R}{L}\right)\frac{d}{dt}\ln L$$

$$+\frac{1}{4}\left(\frac{d}{dt}\ln C\right)^2-\frac{1}{4}\left(\frac{t_{00}R}{L}+\frac{t_{00}G}{C}\right)^2-\frac{1}{2C}\frac{d_2C}{dt^2}.$$

In the case of $x = x_2$ in (4), it follows that

$$y = y_2 = x_2\sqrt{\frac{C(t)}{C(0)}}\exp\left\{\frac{t_{00}}{2}\int_0^t\left[\frac{R(S)}{L(S)}+\frac{G(S)}{C(S)}\right]dt\right\},$$

$$P(t) = \frac{t_{00}}{CL}(1+GR)+\frac{1}{2}\frac{d}{dt}\left(\frac{t_{00}R}{L}-\frac{t_{00}G}{C}\right)+\frac{1}{2}\left(\frac{t_{00}R}{L}-\frac{t_{00}G}{C}\right)\frac{d}{dt}\ln C$$

$$+\frac{1}{4}\left(\frac{d}{dt}\ln C\right)^2-\frac{1}{4}\left(\frac{t_{00}R}{L}+\frac{t_{00}G}{C}\right)^2-\frac{1}{2C}\frac{d^2C}{dt^2}.$$

Equation (3) can be reduced to a vector equation of a canonical type analogous to (5) by carrying out the following substitution:

$$x = z\exp\left\{\frac{1}{2}\int_0^t\left[a_{11}(t)+a_{22}(t)\right]dt\right\}.$$

We obtain

(6) $$\frac{d}{dt}z = B(t)z\text{, where}$$

$$z = \mathrm{colon}(z_1, z_2);\ B(t) = \left\{b_{ij}(t)\right\},\ i, j = 1,2;$$

$$b_{11} = -b_{22} = \frac{1}{2}(a_{11}-a_{22});\ b_{12} = a_{12};\ b_{21} = a_{21}$$

The condition that $\mathrm{Sp}B(t)=0$ is indicative of the canonical character of equation (6). In the case under consideration it is in a rather simple form.

If we introduce a Hamiltonian function, i.e. square form of the system

(7) $$H(t,z_1,z_2)=\frac{1}{2}b_{21}(t)z_1^2-b_{11}(t)z_1z_2-\frac{1}{2}b_{12}(t)z_2^2.$$

Equation (6) can be written in the form of the following system

(8) $$\frac{\partial z_1}{\partial t}=\frac{\partial H}{\partial z_2},\frac{\partial z_2}{\partial t}=\frac{\partial H}{\partial z_1}.$$

Let us compare the canonical system (6) with the equation of a general type (3). It is evident that the elements of the second diagonal of the matrix are identical, while the elements of the main diagonals differ. They are presented in an extended form:

$$b_{11}=-t_{00}\frac{G_{eqv}}{C_2},G_{eqv}=\tfrac{1}{2}\left(G-\frac{R}{\rho^2}\right),b_{22}=-t_{00}\frac{R_{eqv}}{L}$$

$$R_{eqv}=-\rho^2G_{eqv},\rho=\sqrt{\frac{L(t)}{C(t)}}.$$

Hence the conclusion that the necessary and sufficient condition for the oscillating circuit to be described by a canonical equation is

$$R_{eqv}=-\rho^2G_{eqv}.$$

It is obvious that any conservative oscillating circuit ($R = 0$, $G = 0$) is described by a canonical vector differential equation.

It is not difficult to verify that if a Hamiltonian oscillating circuit is described by equation (8), the coefficients of the first derivative has an average value equal to zero, since

$$a_1=\frac{d}{dt}\ln L \text{ or } a_1=\frac{d}{dt}\ln C.$$

Let us express the Hamiltonian function of system (8) by using the circuit parameters

(9) $$H(t,z_1,z_2)=\frac{t_{00}}{2}\left(\frac{q_{00}}{\Phi_{00}C}z_1^2+\frac{\rho G-\dfrac{R}{\rho}}{\sqrt{LC}}z_1z_2+\frac{\Phi_{00}}{q_{00}L}z_2^2\right).$$

When analyzing the canonical system it is important to identify the non-negative condition, $H(t,z_1,z_2)\geq 0$, for all values of the arguments, for which the solution of the equation z_1 and z_2 are unknown. The bracketed middle term in (9) is a serious impediment to such an estimate. That is why we use the inequality $-\frac{z_1^2+z_2^2}{2}\leq z_1z_2<\frac{z_1^2+z_1^2}{2}$ to obtain the following bilateral estimate

(10) $$\frac{t_{00}}{2}\left(1-\frac{1}{2}\left|\rho G-\frac{R}{\rho}\right|\right)\left(\frac{q_{00}}{\Phi_{00}C}z_1^2+\frac{\Phi_{00}}{q_{00}L}z_2^2\right)\leq H(t,z_1,z_2)$$

$$\leq\frac{t_{00}}{2}\left(1+\frac{1}{2}\left|\rho G-\frac{R}{\rho}\right|\right)\left(\frac{q_{00}}{\Phi_{00}C}z_1^2+\frac{\Phi_{00}}{q_{00}L}z_2^2\right).$$

As a result off these inequalities, in the absence of any dissipative losses $(G\equiv 0, R\equiv 0)$, the following equation is obtained

$$H(t,z_1z_2)=\frac{t_{00}}{2}\left(\frac{q_{00}}{\Phi_{00}C}z_1^2+\frac{\Phi_{00}}{q_{00}L}z_2^2\right).$$

This is the normalized instantaneous energy accumulated in the reactances of the oscillating circuit. Canonical systems and their equivalent equations can be either stable or unstable, but they cannot be asymptotically stable.

Let us consider some other forms of the canonical vector system (4), for example the following one:

(11) $$\frac{d}{dt}z=JHz,\quad J=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad H=\begin{pmatrix}b_{21}&-b_{11}\\-b_{11}&b_{12}\end{pmatrix}$$

$$= \begin{pmatrix} \frac{t_{00} q_{00}}{\Phi_{00} C(t)} & \frac{t_{00}}{2}\left[\frac{G(t)}{C(t)} - \frac{R(t)}{L(t)}\right] \\ \frac{t_{00}}{2}\left[\frac{G(t)}{C(t)} - \frac{R(t)}{L(t)}\right] & \frac{t_{00} \Phi_{00}}{q_{00} L(t)} \end{pmatrix}.$$

H is a symmetrical matrix of the Hamiltonian function (9).

Since $J^{-1} = -J$, equation (11) can also be written in the form

$$J \frac{d}{dt} z = Hz \, . \tag{12}$$

Sometimes it is convenient to present the initial system (3) in the form of an integral system of equations. If, for instance, $x(t) = \{x_{ij}(t)\}$, $i,j = 1,2$ is a fundamental system of solutions to (3), it will satisfy the integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} A(S) x(S) dS \, ,$$

where $x(t_0)$ is the matrix of the initial conditions.

This form of the oscillating circuit equation is convenient when using the successive approximations method, in the case of seeking a solution in the form of $x(t) = x(t_0) + \sum_{k=1}^{\infty} x_k(t)$, where $x_k(t) = \int_{t_0}^{t} A(S) x_{k-1}(S) dS$, $k=1,2,..,$ $x_0 = x(t_0)$.

This recurrent formula allows of consistently identifying all series terms in the solution. It would not be difficult to show that this series is absolutely and evenly convergent over an arbitrary finite interval.

Further on we shall demonstrate a multistage transformation approach. If the variables in (3) are substituted in the following way

$$x = B(t) y, \quad B(t) = \begin{pmatrix} \frac{1}{a_{21}(t)} & 0 \\ 0 & 1 \end{pmatrix} \tag{13}$$

the equation will assume the form

(14) $$\frac{d}{dt}y = C(t)y,$$

where $C(t) = \begin{pmatrix} C_{11}(t) & a_{12}(t)\,a_{21}(t) \\ 1 & a_{22}(t) \end{pmatrix}, \quad C_{11} = a_{11} + \left(\frac{d}{dt}a_{21}\right)\frac{1}{a_{21}}.$

A comparsion between (14) and (3) shows that equation(14) describes an oscillating circuit with constant capacitance or constant inductance, while all other respective parameters vary in time.

Let us continue the transformation by performing the following substitution in (14)

$$y = z\exp\left\{\frac{1}{2}\int_{t_0}^{t}[C_{11}(S) + a_{22}(S)]dS\right\}.$$

The result is

(15) $$\frac{d}{dt}z = D(t)z,$$

where $D(t) = \begin{pmatrix} d_{11}(t) & a_{12}(t)\,a_{21}(t) \\ 1 & -d_{11}(t) \end{pmatrix}, \quad d_{11}(t) = \frac{1}{2}\left(a_{11} - a_{22} + \frac{1}{a_{21}}\frac{da_{21}}{dt}\right).$

Obviously (15) is a simplified canonical system, since one of the matrix elements is constant.

Finally, if we carry out the following substitution in equation (15)

$z(t) = R(t)U(t), \quad R(t) = \begin{pmatrix} d_{11}(t) & 1 \\ 1 & 0 \end{pmatrix}$ we shall obtain

(16) $$\frac{d}{dt}U = M(t)U, \quad M(t) = \begin{pmatrix} 0 & 1 \\ r(t) & 0 \end{pmatrix}$$

where $r(t) = a_{12}a_{21} + d_{11}^2 - \frac{dd_{11}}{dt}$.

System (16) describes an oscillating circuit without losses, which contains one time-dependent reactance.

The system (16) is much simpler than the initial system (3), and the solutions of both equations are related in the following way:

(17) $$x(t) = k(t,t_0)N(t)U,$$

where $k(t,t_0) = \exp\left\{\frac{1}{2}\int_{t_0}^{t}\left[a_{11}(\tau) + a_{22}(\tau) + \frac{da_{21}(\tau)}{d\tau}\frac{1}{a_{21}(\tau)}\right]d\tau\right\}$,

$$N(t) = \begin{pmatrix} \frac{1}{2a_{21}}\left(a_{11} - a_{22} + \frac{1}{a_{21}}\frac{da_{21}}{dt}\right) & \frac{1}{a_{21}} \\ 1 & 0 \end{pmatrix}.$$

The investigation of the free processes in linear oscillating circuits with time dependent parameters of a general type is connected with enormous mathematical difficulties conditioned by the general character of the problem. Indeed, it is necessary to examine a huge number of equations (3) characterized by four functions - the elements of the matrix a_{ij}, $i,j=1,2$.

The canonical systems lend themselves to a sufficiently accurate and clear classification by boundedness or boundlessness of their solutions. The possible canonical systems make up a set that can be visualized as a set of points in a three-dimensional cylindric space. The cylindric space breaks down into a set of alternating areas of stability and instability. Certain investigations have been carried out and they have yielded a sufficiently comprehensive picture of the general properties of the free processes in a linear oscillating circuit with periodic parameters.

Classification of the free processes in Hamiltonian oscillating circuits

The canonical systems lend themselves to a sufficiently distinct and clear classification by boundedness or boundlessness of their solutions. It is possible to create a comprehensive picture of the general regularities governing the free processes in a linear oscillating circuit with periodic parameters. For the sake of visualization we shall use a three-dimensional cylindric space which is broken down into a set of alternating areas of stability and instability.

Consider a canonical system with periodic coefficients in the following vector form

$$\frac{d}{dt}z = JHz\,. \tag{18}$$

The fundamental matrix of the solutions of the vector equation (18), given the initial conditions $z(0) = I$, where I is a unit matrix, is presented as

$$z(t) = P(t)e^{Kt}\,. \tag{19}$$

The properties of the matrices in (19) are as follows:

a) $P(t)$ is a real matrix-function, which is either periodic, i.e. $P(t+T) = P(t)$ or antiperiodic, $P(t+T) = -P(t)$, T - period, $P(0) = I$, $\frac{d}{dt}P$ is piece - wise continius, i.e. it exists almost everwhere and is summable;

b) $\mathrm{Det}P(t) \equiv 1$,

c) K is a real square matrix with constant elements, $\mathrm{Sp}K = 0$.

The investigation of the properties of matrix K is of crucial importance in the stability analysis. If the matrix function of the vector equation (18) is known, matrix K will be determined in the following way:

$$K = \frac{1}{T}\ln\left[\pm z(T)\right]. \tag{20}$$

The alternative sign in (20) is chosen in such a way as to secure a real matrix K.

Let us dwell more elaborately on matrix sets $\{K\}$ and $\{P(t)\}$. For the sake of visualization we shall use geometrical terms.

1. Structure of the set $\{K\}$

We shall present the set of various constant second-order matrices K with zero spur in the following form

$$K = \begin{pmatrix} -x & y-z \\ y+z & x \end{pmatrix},$$

where x, y, z are various values along the axes of the Cartesian coordinate system O*xyz*. This presentation yields an ordinary three-dimensional (Euclidean) space. A completely defined matrix *K* corresponds to each point in this space.

The characteristic equation of the matrix,

$$\mathrm{Det}(K - \lambda I) = 0 \tag{21}$$

has the following form: $\lambda^2 = x^2 + y^2 - z^2$.

Given a real λ, this characteristic equation represents a single-band hyperboloid, whose canonical equation is in the following form:

$$\frac{x^2}{\lambda^2} + \frac{y^2}{\lambda^2} - \frac{z^2}{\lambda^2} = 1.$$

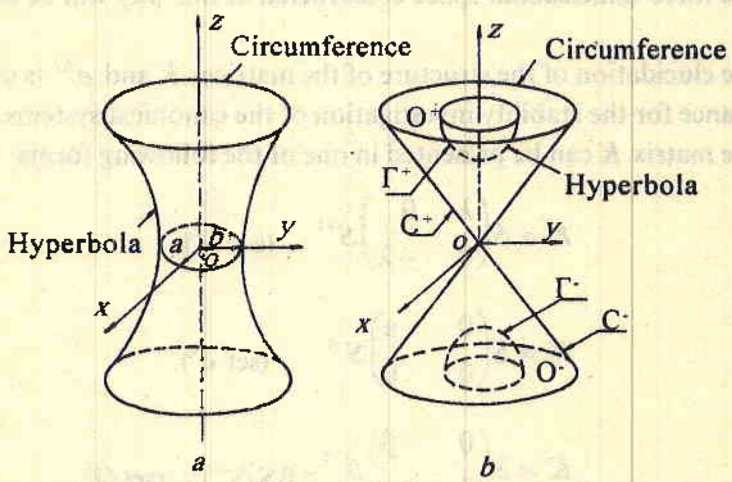

Fig. 2. Single-band hyperboloid illustrated the characteristic equation of the matrix at a real (*a*); double-band hyperboloid illustrated the same at imaginary (*b*)

The graphic representation of the single-band hyperboloid is shown in Fig. 2*a*, where $a = \lambda$ and $b = \lambda$ are the real semi-axes and $c = j\lambda$ is the imaginary semi-axis.

If λ is imaginary, (21) determines a double-band hyperboloid:

$$\frac{x^2}{\lambda^2} + \frac{y^2}{\lambda^2} - \frac{z^2}{\lambda^2} = -1.$$

The graphic representation of the double-band hyperboloid is given in

Fig. 2*b*, where $a = j\lambda$ and $b = j\lambda$ are imaginary semi-axes, and $c = \lambda$ is a real semi-axis.

When $\lambda = 0$, the result is a borderline case of a cone: $x^2 + y^2 = z^2$ separating the two families of the above mentioned hyperboloids. Conventionally, we shall ascribe a plus sign to the cone and hyperboloid tops, and a minus sign to their bottoms. The set of the points on the cone surface will be denoted as C^+ and C^- respectively. The set of the points in the interior of the cone ($x^2 + y^2 < z^2$) will be denoted as $O(O^+$ and $O^-)$ while that of the exterior points ($x^2 + y^2 > z^2$) will be designated as *H*.

It turns out that systems (18) with periodic or antiperiodic solutions correspond to the set *O* (those are systems with bounded solutions), while systems with unbounded solutions are in congruence with the set *H*. We shall denote the surface points of the double-band hyperboloid family as $\Gamma(\Gamma^+$ and $\Gamma^-)$.

The three-dimensional space constructed in this way will be designated as R^3.

The elucidation of the structure of the matrices *K* and e^{Kt} is of particular significance for the stability investigation of the canonical systems.

The matrix *K* can be presented in one of the following forms:

(a) $$K = S\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} S^{-1} \quad (\text{set } H),$$

(b) $$K = S\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix} S^{-1} \quad (\text{set } C),$$

(c) $$K = S\begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix} S^{-1} = \beta SJS^{-1} \quad (\text{set } O),$$

where *S* is a real matrix, λ, ε and β are real numbers.

It follows from equation $S^{-1}e^{Kt}S = e^{S^{-1}KS}$ that:

- when $K \in H$ - $e^{Kt} = S\begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{-\lambda t} \end{pmatrix} S^{-1}$,

- when $K \in C$ - $e^{Kt} = S\begin{pmatrix} 1 & \varepsilon t \\ 0 & 1 \end{pmatrix} S^{-1}$,

- when $K \in O$ - $e^{Kt} = Se^{A\beta t}S^{-1} = S(I\cos\beta t + J\sin\beta t)S^{-1}$.

Hence, in case (a) both solutions of the fundamental system (18) are unbounded, if the whole time axis is taken into account $t \in (-\infty, \infty)$. If the object of consideration is the semi-axis $t \in [0, \infty)$, one of the solutions of the fundamental system should be regarded as unbounded (exponentially increasing), and the other one as bounded (exponentially decreasing).

In case (b) one solution under initial conditions $z_1(0) = S\begin{pmatrix}0\\1\end{pmatrix}$ is periodic or anti-periodic, while the other one - under initial conditions $z_2(0) = S\begin{pmatrix}1\\0\end{pmatrix}$ - is linearly increasing. In case (c) all solutions are bounded.

Let us once again refer to formula (20) and consider the issue of the signal-valuedness of the matrix K.

For the sake of simplifying the presentation, we shall introduce the following denotations: $KT=y$, $z(T)=B$. Then equation (20) takes the following form:

(22) $$e^{y} = \pm B \cdot$$

As a result of a relevant transformation the matrix $\pm B$ can be reduced to one of the following forms:

(a′) $$\begin{pmatrix}\mu & 0\\ & 1\\ 0 & \mu\end{pmatrix}, \ \mu > 0, \ \mu \neq 1,$$

(b′) $$I + \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix},$$

(c′) $$\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}, \ \alpha^2 + \beta^2 = 1.$$

If the matrix $\pm B$ can be reduced to form (a') or (b'), there is a solution only in the case when the characteristic numbers of matrix $\pm B$ are positive. The sign in (22) is chosen on the basis of this condition as well. This is the only solution and it is obtained by using the following formulae:

In case (a')

$$y = \ln(\pm B) = \frac{2\mu \ln \mu}{\mu^2 - 1}\left(\pm B - \frac{\mu^2 + 1}{\mu^2 - 1} \ln \mu I \right) \tag{23}$$

In case (b')

$$\ln(\pm B) = \pm B - I \,. \tag{24}$$

If $B = \pm I$, we arrive at the solution by applying the formula:

$$\ln(\pm I) = n\pi S^{-1} J S \quad (n = 0, \pm 1, \pm 2, \pm \ldots), \tag{25}$$

where n is an even number when a plus sign is ascribed and a negative number when a minus sign is attributed.

When $\pm B$ can be reduced to form (c') and $B \neq \pm I$, all values $\ln(\pm B)$ are yielded by the formula

$$\ln(\pm B) = \frac{\varphi + n\pi}{\sin \varphi} B - (\varphi + n\pi) \operatorname{ctg} \varphi \, I \tag{26}$$

$$(n = 0, \pm 1, \pm 2, \pm \ldots)$$

where $e^{\pm j\varphi}$ are the characteristic numbers of the matrix $B(0 < \varphi < \pi)$; n is an even number when a plus sign is ascribed and a negative number when the sign is minus.

It follows from (22) that $\operatorname{Det} e^y = e^{Spy} = 1$, $Spy = 0$ i.e. all solutions y of equation (22) fall in space R^3.

For matrices y and $\pm B$ it is possible to have simultaneous occurence of either cases (a) and (a'), or cases (b) and (b'), or cases (c) and (c'). So, we can conclude that in the case of unbounded increase in the solution, i.e. (a') and (b'), matrix $K = \frac{1}{T}[\pm z(T)]$ has a single value and $K \in H \cup C^+ \cup C^-$

In the case of a bounded solution (c′) the matrix K is not a single-valued one. There are two options here: a non-trivial one, when $K \in O$, and a trivial one, for $K=O$ (in the second option the matrix degrades to a zero one).

Let us use Γ_0^+ and Γ_0^- to denote the top and bottom of the hyperboloid $x^2+y^2-z^2=-\dfrac{\pi^2}{4T^2}$; O_0^+ and O_0^- to designate respectively the upper and lower area between the cone: $x^2+y^2-z^2=0$ and this hyperboloid; O_1^+ and O_1^- to mark respectively the areas between the top and the bottom of the cone: $x^2+y^2-z^2=0$ and the hyperboloid: $x^2+y^2-z^2=-\dfrac{\pi^2}{T^2}$;

If we take into account the fact that the matrix $P(t)$ has a single value determined by the matrix K in (19), it is obvious that in order to ensure a one-value functional matrix of solutions (19) it would be necessary and sufficient to select the matrix K falling in the area.

2. Structure of set $\{P(t)\}$

We shall use Θ_2 to denote the set of second-order matrices with constant real elements and with a determinant equal to one. Let us assume that x is a matrix belonging to this set. It can always be presented in the form

(27) $$x = a|b,$$

where a is a vector corresponding to the first column of the matrix, and b is a vector corresponding to the second column of the matrix. The elements of matrix x are expressed through the projections of vectors a and b: $x=\begin{pmatrix} a_x & b_x \\ a_y & b_y \end{pmatrix}$.

The angle between the vectors is established according to the formula:

$$\alpha = \arccos\frac{(a,b)}{|a||b|} = \arccos\frac{a_x b_x + a_y b_y}{\sqrt{\left(a_x^2+a_y^2\right)\left(b_x^2+b_y^2\right)}}.$$

It turns out that the determinant of the matrix x can be presented in the form

(28) $$\text{Det } x = |a||b| \sin\alpha = 1 .$$

Let us examine solution (19) of the canonical system of two equations. Given a fixed t, the matrix $z(t)$ is an element of the set Θ_2; in the case of changing time this matrix is continuously converted from one element (matrix) of the set Θ_2, into another element of the same set, i.e. a trajectory is described in Θ_2, which can be visualized as a curve situated within a torus (Fig. 3). We are interested in the normalized matrix of the solutions when the initial point of the trajectory is an unit matrix, and the final one is the matrix $z(t)$.

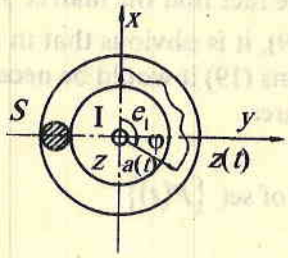

Fig. 3 Visualization of the solution as a curve situated within a torus

Fig. 3 can be treated in another way as well. Let us consider a section of the torus with plane $z = 0$. The result is a circuit in the plane Oxy. Each point on this circuit corresponds to a constant vector $a = \begin{pmatrix} x \\ y \end{pmatrix}$. Hence it is obvious that Fig. 3 shows one of the solutions $z(t)$, corresponding to the initial condition $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. φ is the angle of rotation of the vector $x(t)$ for the time interval $[0, t)$. The angle of rotation over the time span of one period T, i.e. for the interval $[t, t = T]$, is of particular interest.

It follows from (19) that $z(T) = P(T)a, a$ is a constant vector.

As a result of the condition that $P(t+T) = \pm P(t)$, over the time interval of $\Delta t = T$ the vector $z(t)$ turns at an angle divisible by π, i.e.

$$\varphi_T = n\pi, \quad (n = 0, \pm 1, \pm 2, \ldots) \tag{29}$$

where n is an even number for a periodic matrix $P(t)$ and an odd number for an anti-periodic matrix. We have to point out that n does not depend on the initial conditions, i.e. two arbitrary vector solutions belonging to the relevant canonical system rotate in synchronism so that the angle between them remains unchanged. Hence, n in (29) is a system characteristic. Moreover, it proves to be quite an important characteristic from the viewpoint of the stability theory.

Formula (29) allows of classifying canonical systems according to the values of n. Indeed, each particular canonical system is characterized by its own matrix $z(t)$, hence its own matrix as well, determining the number of semi-revolutions n_s over an interval $\Delta t = T$ for an arbitrary vector-solution of the canonical system under consideration.

Let us carry out a qualitative examination of the behaviour of the solutions of canonical systems. Let us assume that there is a canonical system

$$\frac{d}{dt}z = A(t)z, \quad SpA(t) = 0. \tag{30}$$

We shall consider the set L^3, which is made up of triads of periodic piecewise continuous functions (elements of matrix $A(t)$) $a_{11}(t), a_{12}(t), a_{21}(t)$. We introduce the following norm in L^3

$$\int_0^T \left[|a_{11}(t)| + |a_{12}(t)| + |a_{21}(t)| \right] dt \,.$$

Then L^3 is transformed into a complete linear normalized space (Banach space).

Each matrix A has its corresponding matrix function $z(t)$ which, in its turn, is juxtaposed to a couple of matrices $P(t), K$ where $P(t) \in \Omega, \Omega$ is metric space, $K \in H \bigcup C \bigcup O_1^+$ The correspondence indicated above is written in the following way:

$$L^3 = \Omega \times \left(H \bigcup C \bigcup O_1^+ \right).$$

We use o to denote the beginning of the coordinate system in R^3. Then $C = C^+ \bigcup o \bigcup C^-$. We break down space Ω into a countable set of subspaces,

$\Omega = \sum_{n=-\infty}^{\infty} \Omega_n$, and introduce the following designations:

$$H_n = \Omega_n \times H, \qquad C_n^{*+} = \Omega_n \times C^+,$$
$$O_n = \Omega_n \times O_1^+, \qquad C_n^{*-} = \Omega_n \times C^-,$$
$$C_n = \Omega_n \times C, \qquad C_n^{*0} = \Omega_n \times O$$

The product of the sets should be seen in the following way: $A(t) \in H_n$, if $P(t) \in \Omega$, $K \in H$.

The following denotations are used:

$$H = \bigcup_{n=-\infty}^{\infty} H_n, O = \bigcup_{n=-\infty}^{\infty} O_n, C = \bigcup_{n=-\infty}^{\infty} C_n, C^{*\pm} = \bigcup_{n=-\infty}^{\infty} C_n^{\pm}, C^{*0} = \bigcup_{n=-\infty}^{\infty} C_n^0, C^* = C^+ \cup C^-,$$

The fundamental matrices $z(t)$ related to the canonical systems belonging to different areas of stability have different representations:

$$A(t) \in H \to z(t) = P(t)S\begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{-\lambda t} \end{pmatrix} S^{-1} \qquad (\lambda > 0),$$

$$A(t) \in O \to z(t) = P(t)S\begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix} S^{-1} \qquad \left(0 < \beta < \frac{1}{T}\right),$$

$$A(t) \in C^{*\pm} \to z(t) = P(t)S\begin{pmatrix} 1 & \pm t \\ 0 & 1 \end{pmatrix} S^{-1},$$

The open connected sets O_n are referred to as stability (boundedness) areas, the open linked sets H_n are respectively termed areas of instability, C_n are the boundaries between the areas of stability and instability (the boundaries themselves are referred to the unstable area, in accordance with another law - that of the growing unbounded solutions).

Let us consider the issue of the composition of space L^3, which is important for applied problems. The matrix function of the fundamental system (19) can be viewed as a trajectory in torus Θ_2. Each canonical system has its corresponding matrix function. Therefore one intuitively arrives at the assertion that the set of Hamiltonian systems (30), or the set of matrices $\{A(t)\}$ which is the same, has a corresponding set of matrix functions $\{z(t)\}$. For the purpose of proving this correspondence, set $\{z(t)\}$ should be arranged in such a way as to form a space analogous to space L^3 for matrix $\{A(t)\}$. It should also be shown that the correspondence $A(t) \leftrightarrow z(t)$ is reciprocally continuous. Seeking to arrange set $\{z(t)\}$, we introduce the following intervals:

$$\rho(z_1, z_2) = \sup_{0 \le t \le T} \| z_1(t) - z_2(t) \| + \int_0^T \left\| \tfrac{d}{dt}\left[z_1(t) - z_2(t) \right] \right\| dt .$$

Then set $\{z(t)\}$ (like L^3 for $\{A(t)\}$) becomes a complete linear normalized space. These two spaces can be treated as identical and denoted in the same way - as L^3. The dependence of the matrix functions $z(t)$ on the matrix $A(t)$ is continuous. Moreover there is observance of this continuity both by norm and by interval.

As shown above, the points on torus Θ_2 are unimodular matrices (matrices with a determinant equal to one). The points in the space R^3 for matrices K represent matrices with a trace equal to zero. For the purpose of achieving a single value it would be sufficient to select such points for matrices K that lie between the upper and lower bottom of the two-band hyperboloid: $z^2 - x^2 - y^2 = \frac{\pi^2}{4T^2}$, and to include one of the two boundaries in this part of space R^3. The bottoms of the two-band cone: $x^2 + y^2 = z^2$, are situated between the bottoms of this hyperboloid. The space between the cone bottoms corresponds to the unstable area. The space between the respective bottoms of the cone and the hyperboloid corresponds to the stable area. The surface of the cone is the boundary between the stable area and the unstable one, but it is treated as belonging to the unstable area. Any point belonging to torus Θ_2, i.e. any unimodular matrix with constant parameters z can always be presented in the form: $z = \pm e^{Kt}$ In this way the stable and unstable areas, which are already determined by matrix K, can be transferred to torus Θ_2. It has already been shown that the matrix function $z(t)$ of system (30) "winds" round the torus Θ_2, and the number of windings for a time interval of $\Delta t = T$ is $\frac{n}{2}$ (when the canonical system is in the n-th stable or unstable range). In the case of stability, the end of the winding at the close of interval $[t, t = T]$ proves to be at the same distance from the torus center as in the beginning of the interval. In the case of instability, the end of the winding proves to be further away from the torus center than its beginning. Let us replace the torus containing $\frac{n}{2}$ windings with $\frac{n}{2}$ identical tori, each containing one winding of the solution. Let us cut each torus at one and the same place along its section and then shift the ends of the section in different direction along axis z (the result is a spiral winding). Then we stick together the section ends of the different tori so as to link up the solution windings and obtain the complete solution. Thus, instead of $\frac{n}{2}$ tori, the result is a spiral area with $\frac{n}{2}$ windings. By deforming the spiral area directly one can obtain a cylindrical space whose section is shown in Fig. 4. The space itself can be obtained by rotating the figure round the hatch-dotted line.

The cylindric space R^3 encompasses various canonical systems. A particular case of such systems is, for instance, Mathieu's equation (the dotted line in Fig. 4 indicates the area corresponding to Mathieu's equation).

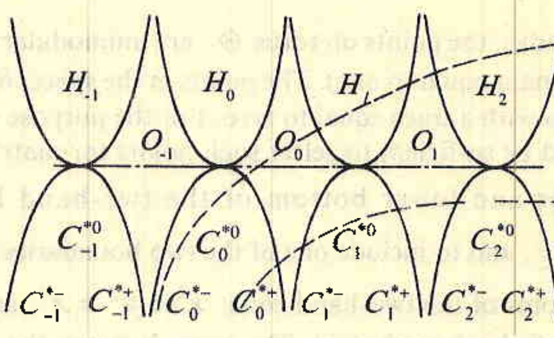

Fig. 4. Three-dimensional cylindrical space the points of which represent the set of canonical equations of the second order

Conclusion

The systems of linear differential equations with periodic coefficients can not be solved in a general form. This is why the qualitative methods for solving those equations are of special interest. With that end in view it is necessary the initial system of equations to be transformed to the most appropriate form. It turns out to be that for different purposes of the analysis "the most simple form of the equations" are quite different in different cases. It is not possible to find such form of equations allowing to tackle them in a general form. Herewith the resonable areas of applying different form of equations have been discussed. It has been demonstrated that the system of equations, describing an oscillating system, can be transformed using variables substituting, in the form, characteristic for the Hamiltonian systems. A qualitative picture of the free processes in an oscillating system with periodic in time parameters has been presented on the basis of the mathematical theory of the Hamiltonian systems. Special attention has been payed to the problem of stability according to Lyapunov's statements. A three-dimensional cylindrical space has been put in correspondancet the set of equations describing every possible oscillating systems with periodic parameters. The space has been divided into accounting areas, corresponding to the areas of stability and unstability of the canonical systems. A concrete oscillating system with periodic parameters is set in correspondence to the every point of this space. Such an approach allows to make a methodologically consistent classification of the oscillating circuits with periodical parameters in accordance with the most important indication, namely the stability and unstability according to Lyapunov's propoundings.

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Качествен анализ на свободните процеси в обобщен линеен трептящ кръг с периодични параметри. I. Структура на диференциалните уравнения и класификация на свободните процеси в Хамилтонови трептящи кръгове

Николай Бирюк, Владимир Дамгов

(Резюме)

Системите линейни диференциални уравнения с периодични коефициенти не се решават в общ вид, ето защо особено значение придобиват качествените методи за изследване. За целта е необходимо изходната система диференциални уравнения да се преобразува към най-подходящата форма. В статията се обсъждат целесъобразните области на приложение на една или друга форма на системата диференциални уравнения.

Разгледан е обобщен трептящ кръг, съдържащ променливи във времето реактивни (C и L) и активни елементи (R и G). Показва се, че системата диференциални уравнения, описваща обобщения трептящ кръг, може чрез замяна на променливите да се приведе към вид, характерен за Хамилтоновите системи. Използвайки математическата теория на Хамилтоновите системи, е дадена качествена картина на свободните процеси в трептящ кръг с периодични във времето параметри. Особено внимание е отделено на задачата за устойчивост по Ляпунов. На множеството уравнения на всевъзможни трептящи кръгове с периодични параметри е поставено в съответствие тримерно цилиндрично пространство. Това пространство е разбито на броимо множество области, съответстващи на областите на устойчивост и неустойчивост на каноничните системи. На всяка точка от това пространство се поставя в съответствие конкретен кръг с периодични параметри. Такъв подход позволява да се направи методологически издържана класификация на трептящите кръгове по най-важния признак - устойчивостта или неустойчивостта по Ляпунов.