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Excitation of „quantized“ oscillations under external inhomogeneous action[1]

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Introduction

A modelling system and oscillation excitation mechanism are presented that might find application in revealing the generation mechanisms of planetary magnetosphere radio sources and the wave interaction mechanisms in the Earth ionosphere and magnetosphere as well as the excitation of VLF waves in the near-Earth space.

The phenomenon of continuous oscillations excitation with ampltude from descrete value set of stationary amplitudes is demonstrated on the basis of a common model — an oscillating system under the action of external periodic force, nonlinear regarding the excited system coordinates. The phenomenon includes as particular case cyclotron process of charged particles acceleration. The phenomenon manifests itself in oscillating systems under the inhomogeneous action of external periodic forces.

The Nonlinear Theory of oscillations consider mainly the action of periodic forces which do not depend on the coordinates or are linear with respect to coordinates of excited systems (the classical parametric systems) [1, 2, 3]. During the last years, linear excited parametric resonance in the presence of a quadratic, cubic or periodic nonlinearity has been investigated [4].

The paper deals with the phenomenon of oscillation excitation under the action of an external nonlinear HF force, which is nonlinear as regards the coordinate of the excited system [5, 6]. Such system may be considered as autooscillating system with external power supply [7]. The investigation is motivated by servey the known from SHF and physical electronics, radiophysics, mechanics, technics of charged particle acceleration, processes and phenomena in plasma and other medium based on the inertia properties of the particles and inhomogeneous interactions etc. [1—7, 8, 9, 10], the problem examined by Fermi, to be known as a possible cosmic ray acceleration mechanism when char-

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ged particles are accelerated by collisions with moving magnetic field structures [11]. In every particular case and mode the interaction mechanism has been revealed differently — self-modulation, grouping, phase selection etc. However all these mechanisms are based on a common principle: the HF external force acts nonlinearly as regards the particles motion coordinates. In the present work it is shown, that the mechanism of LF oscillation excitation with discrete set of possible stable amplitudes is connected with phase capture and dynamical phase adaption, providing the necessary energy contribution to the oscillations during the external inhomogeneous influence. References as LF and HF are used only relatively. In the common case, the phenomenon is manifested in all frequency bands in oscillating systems under the action of external HF periodic force, nonlinear to excited systems coordinates. When the excited system and the power supply source interact, force is formed, which is frequency or phase (in general — argument) modulated in character. Characteristic system argument is adaptively tuning phase, providing the most advantageous interaction between the excited oscillation system and the high frequency power supply. Thus, the method of oscillations excitation is called symbolically short "argument method" [5].

The phenomenon of continuous oscillation excitation with amplitude from discrete value set of possible stationary amplitudes is demonstryted analytically for two cases (two analytical conditions) — first, when the nonlinearity of harmonic-force-external action is presented by β-function and the influence is subjected to the lower equilibrium point of the trajectory, and, second, when the external harmonic force acts over a trajectory zone with a finite lenght.

Analysis: the nonlinearity of external harmonic force is presented by δ-function

The motion in different oscillating systems under the action of external periodic force, nonlinear with respect to the system coordinate in general may be described by the following equation:

$$\ddot{x}+2\delta_0\dot{x}+\omega_0^2x+f(x)=F_0(x,\ t_r), \tag{1}$$

where x is the generalized system coordinate, δ_0 is coefficient describing the system dissipative properties, $f(x)$ is function characterizing the excited system nonlinearity, $F_0(x,\ t_r)$ is external periodic force nonlinear to the system coordinate x, t_r is real time.

Taking into account the wide variety of system, described by Eq. (1), for the sake of analysis we select an concretized equation described the pendulum motion. The pendulum is common oscillating model as it is isomorphic to a variety of physical phenomena, particularly such as radio-frequency driven quantum-mechanical Josephson junction, charge density wave transport, cosmic particles in certain conditions etc. [12].

The equation describing pendulum swing caused by the action of a force, nonlinear to the coordinate, can be written in the form

$$\ddot{x}+2\delta_0\dot{x}+\omega_0^2\sin x=F_0(x,\ t), \tag{2}$$

where x is the angular distance to equilibrium, ω_0 is the resonance frequency of the small oscillations, $t=\omega_0t_r$.

In order to integrate the nonlinear Eq. (2) using the methods of the Theory of Nonlinear oscillations, we introduce new variable y and nonlinear time τ. So, the strongly nonlinear reactive term $\sin x$ in Eq. (2) may be excluded. The transformation of variables is performed by the scheme proposed by K. A. Samoylo [13], thus:

(3) $$y = \operatorname{sign} x \sqrt{2 \int_0^x \sin x' dx'} = 2 \sin \frac{x}{2},$$

(4) $$\frac{dt}{d\tau} = \frac{dx}{dy} = \frac{y}{\sin [x(y)]} = G(y).$$

Functions $x(y)$ and $G(y)$ in Eq. (4) are easily expressed, taking jnto account Expr. (3):

(5) $$x(y) = 2 \arcsin\left(\frac{y}{2}\right), \quad G(y) = \frac{1}{\sqrt{1 - \frac{y^2}{4}}}.$$

Substituting Exprs. (3) and (4) in Eq. (2) we obtain

(6) $$\frac{d^2y}{d\tau^2} + \beta^2 y = -2\delta_d \frac{dy}{d\tau} + F(x, \tau) G(y) + (\beta^2 - 1) y,$$

where $2\delta_d = \frac{2\delta_0}{\omega_0}$ and $F(x, \tau) = \frac{F_0(x, t)}{\omega_0^2}$, $(\beta^2 - 1)$ corresponds to the frequency detuning, $\beta \sim 1$.

The transition to new variables makes the system quite similar to a linear concervative system, whose state is represented by a point, moving jn phase space on a circle with constant angular velocity. For such a system, common methods of Nonlinear oscillation theory can be applied. It should be mentioned that in terms of the new valiables all initial system features are kept. Transformations (4) and (5) are appropriate if conditions $G(0) = 1$ and $G(y) > 0$ for all y values are fulfilled. Obviously Condition 1 is satisfied (see Exp. (5)). Condition 2 is fulfilled for $-\pi < x < \pi$ or $-2 < y < 2$. Further consideration will be performed for this y values interval. Physically it means, that initial conditions and external action provide pendulum swing with angle amplitude less than $\pm\pi$.

We assume that solution of Eq. (6) is:

(7) $$y = R \cos \Psi = R \cos (\beta\tau - \varphi_v),$$

where R and φ_v are oscillations amplitude and phase.

The dependence of normalized time t on angle Ψ can be expressed in agreement with Exprs. (4), (5) and (7) as

(8) $$t = \frac{1}{\beta} \int_0^{\Psi} \frac{d\psi}{\sqrt{1 - \frac{R^2}{4} \cos^2 \Psi}}.$$

Considering Expr. (8), the normalized oscillations period is:

(9) $$T_0 = \frac{1}{\beta} \int_0^{2\pi} \frac{d\Psi}{\sqrt{1 - \frac{R^2}{4} \cos^2 \Psi}} = \frac{4}{\beta} K\left(\frac{R}{2}\right),$$

where $K\left(\frac{R}{2}\right)$ is the full elliptic integral of first kind.

The shortened (averaged) differential equations [1, 2, 3, 13] for amplitude R and phase φ_ν can be written as:

(10a) $$\left\langle\frac{dR}{d\tau}\right\rangle=-\frac{1}{2\pi\beta}\int_0^{2\pi} L\sin\Psi d\Psi,$$

(10b) $$\left\langle\frac{d\varphi_\nu}{d\tau}\right\rangle=-\frac{1}{2\pi\beta R}\int_0^{2\pi} L\cos\Psi d\Psi,$$

where the sign $\langle\ \rangle$ denotes the procedure of averaging by time τ,

$$L=2\delta_d\beta R\sin\Psi+F(x,\ \tau)G(y)+(\beta^2-1)R\cos\Psi.$$

Taking into account that

$$\int_0^{2\pi}\sin^2\Psi G(y)d\Psi=\int_0^{2\pi}\frac{\sin^2\Psi}{\sqrt{1-\frac{R^2}{4}\cos^2\Psi}}d\Psi=4K\left(\frac{R}{2}\right)+\frac{16}{R^2}\left[E\left(\frac{R}{2}\right)-K\left(\frac{R}{2}\right)\right],$$

where $E(.)$ is the full elliptic integral of second kind, the shortened equations (10) take the form

(11a) $$\left\langle\frac{dR}{d\tau}\right\rangle=-\frac{1}{\pi}\beta R\left\{4K\left(\frac{R}{2}\right)+\frac{16}{R^2}\left[E\left(\frac{R}{2}\right)-K\left(\frac{R}{2}\right)\right]\right\}-\frac{1}{2\pi\beta}\int_0^{2\pi}F(x,\ \tau)G(y)\sin\Psi d\Psi,$$

(11b) $$\left\langle\frac{d\varphi_\nu}{d\tau}\right\rangle=-\frac{1}{2\pi\beta R}\int_0^{2\pi}F(x,\ \tau)G(y)\cos\Psi d\Psi-\frac{\beta^2-1}{2\beta}.$$

Now, let us concretize the function $F_0(x,\ t)$ as follows:

(12) $$F_0(x,\ t)=\delta(x)P\sin\nu t,$$

where $\delta(x)$ — δ-function, P and ν are the external harmonic force amplitude and frequency correspondingly. We assume that $\nu=N\omega_0$, where $N=1, 2, 3, \ldots$ Taking into account the solution form (7), δ-function $\delta(x)$ can be presented in the form

(13) $$\delta(x)=\sum_i\left|\frac{d\Psi}{dx}\right|\delta(\Psi-\Psi_{0,i}),$$

where the values $\Psi_{0,i}$ are determined by the equation

(14) $$x(\Psi_{0,i})=0.$$

Considering equations (13) and (14) the equations (11) become

(15a) $\langle \frac{dR}{d\tau} \rangle = -\frac{1}{\pi}\beta R\left\{4K\left(\frac{R}{2}\right)+\frac{16}{R^2}\left[E\left(\frac{R}{2}\right)-K\left(\frac{R}{2}\right)\right]\right.$
$-\frac{1}{2\pi\beta}\left[GP\sin\nu t\left(\frac{\pi}{2}\right)\left|\frac{d\Psi}{dx}\right|_{\frac{\pi}{2}} - GP\sin\nu t\left(\frac{3\pi}{2}\right)\left|\frac{d\Psi}{dx}\right|_{\frac{3\pi}{2}}\right],$

(15b) $\langle \frac{d\varphi_\nu}{d\tau} \rangle = -\frac{\beta^2-1}{2\beta}.$

Noting that $\frac{d\Psi}{dx}=\frac{d\Psi}{d\tau}\frac{d\tau}{dt}\frac{dt}{dx}=\frac{d\Psi}{d\tau}\frac{d\tau}{dt}\frac{d\tau}{dy}=-\frac{1}{G(y)R\sin\Psi}$ and that $G\left(\frac{\pi}{2}\right)=G\left(\frac{3\pi}{2}\right)=1$, the Eq. (15a) can be rewritten as

(16) $\langle \frac{dR}{d\tau} \rangle = -\frac{1}{\pi}\beta R\left\{4K\left(\frac{R}{2}\right)+\frac{16}{R^2}\left[E\left(\frac{R}{2}\right)-K\left(\frac{R}{2}\right)\right]\right.$
$\left.-\frac{P}{2\pi\beta R}\left[\sin\nu t\left(\frac{\pi}{2}\right)-\sin\nu t\left(\frac{3\pi}{2}\right)\right]\right\}.$

From Eq. (9) we obtain

(17) $$\beta=\frac{2\nu K\left(\frac{R}{2}\right)}{\pi N}.$$

Introducing the designation $t\left(\frac{\pi}{2}\right)=t_1$ and taking into account Exprs. (9) and (17) we can write

$$t\left(\frac{3\pi}{2}\right)=t_1+\frac{2}{\beta}K\left(\frac{R}{2}\right),\quad \sin\nu t\left(\frac{3\pi}{2}\right)=(-1)^N\sin\nu t_1.$$

Let us now consider two cases:

a) Case of even N ($N=2l$, $l=1, 2, 3, \ldots$).

In this case $\sin\nu t\left(\frac{\pi}{2}\right)-\sin\nu t\left(\frac{3\pi}{2}\right)=0$ and there are no stationary solution (the oscillations are demped);

b) Case of odd N ($N=2l+1$, $l=0, 1, 2, 3, \ldots$).

In this case $\sin\nu t\left(\frac{\pi}{2}\right)-\sin\nu t\left(\frac{3\pi}{2}\right)=2\sin\nu t_1$ and

$$\left|\begin{array}{l}\langle \frac{dR}{d\tau} \rangle = e(R,\varphi_\nu),\\ \langle \frac{d\varphi_\nu}{d\tau} \rangle = g(R,\varphi_\nu),\end{array}\right.$$

where

(18a) $$e(R,\varphi_\nu)=-\frac{1}{\pi}\beta R\left\{4K\left(\frac{R}{2}\right)+\frac{16}{R^2}\left[E\left(\frac{R}{2}\right)-K\left(\frac{R}{2}\right)\right]-\frac{P}{\pi\beta R}\sin\nu t_1,\right.$$

(18b) $$g(R,\varphi_\nu)=-\frac{\beta^2-1}{2\beta}.$$

For stationary mode ($e(R,\varphi_\nu)=0$ and $g(R,\varphi_\nu)=0$) from Eq. (18b) find the condition $\beta=1$, which can be rewritten considering Expr. (17) as $K\left(\frac{R}{2}\right)=\frac{\pi}{\nu}\times\left(l+\frac{1}{2}\right)$, $l=0, 1, 2, 3, \ldots$

Denoting $k = \frac{R}{2}$ (the module of the elliptic function), from Eq. (18a) we can find the second condition of stationary mode in the form

(19) $$\frac{16\delta_d}{P}[E(k)-(1-k^2)K(k)]-\sin \nu t_1=0.$$

When $k \to 0$ the Eq. (19) is simplified

(20) $$\frac{4\pi\delta_d k^2}{P}+\sin \nu t_1=0,$$

and corresponds to the condition

(21) $$|P|>4\pi\delta_k k^2.$$

For the sake of stability estimation we can rewrite Eqs. (18) under the condition $k \to 0$ as

(22a) $$e(R, \varphi_\nu) = -2\delta_d k - \frac{P}{2\pi\beta k}\sin \nu t_1,$$

(22b) $$g(R, \varphi_\nu) = -\frac{\beta^2-1}{2\beta},$$

where $\beta=1$.

The characteristic equation can be written as

$$\lambda^2-\lambda(e_R+g_\varphi)+e_R g_\varphi-e_R g_R=0,$$

taking the final form

(23) $$\lambda(\lambda-e_R)=0,$$

where e_R, g_φ, g_R are the corresponding partial derivatives.

From Eq. (23) we find $\lambda_1=0$, $\lambda_2=e_R$. The stability condition is: $\lambda_2=e_R<0$ i. e. $e_k<0$.

Using Eq. (22a) we obtain $e_k=-2\delta_d+\frac{P}{2\pi k^2}\sin \nu t_1$.

Comparison with (20) reveals the stability condition in the form

(24) $$e_k=-4\delta_d<0.$$

As the value $\delta_d>0$ *apriori*, the inequality (24) is fulfilled and the solution for odd N describes discrete set of stable stationary oscillitions.

Analysis: The external harmonic force acts over a trajectory zone with a finite length

We consider the equation, describing pendulum motion, under nonhomogeneous action of external harmonic force, in the form

(25) $$\ddot{x}+2\delta_d\dot{x}+\sin x=\varepsilon(x)P\sin \nu t,$$

where

$$\varepsilon(x)=\begin{cases}1, & \text{when } |x|\le d,\ d<1\\ 0, & \text{when } |x|>d\end{cases}$$

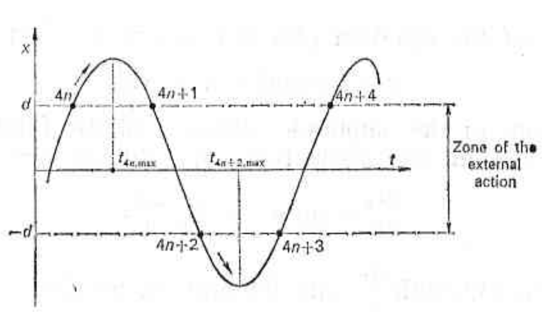

Fig. 1

determine the trajectory zone of the external influence.

Conditionally, we number the time moments, determined by the zone of action, as it is shown in Fig. 1.

The pendulum motion in the time intervals $[4n, 4n+1]$, $[4n+2, 4n+3], \ldots$ (out of the action zone) can be described by unperturbed equation

(26) $$\frac{d^2x}{dt^2}+\sin x=0.$$

Multiplying Eq. (26) with $\frac{dx}{dt}$ and integrating, we find

(27) $$\frac{1}{2}\left(\frac{dx}{dt}\right)^2-\cos x=W-1,$$

where W is an integration constant corresponding to the full system energy.

From Eq. (27) we obtain

(28) $$\frac{dx}{dt}=\pm\sqrt{2W-4\sin^2\frac{x}{2}}\,.$$

Introducing the designation $u=\frac{x}{2}$ and $\sin u=z$ and considering Eq. (28), we can write

$$t-\alpha=\int\frac{dz}{\pm\sqrt{(1-z^2)\left(\frac{W}{2}-z^2\right)}}, \text{ where } \alpha\text{-constant.}$$

Further on we use the incomplete normal elliptic integral of first kind $F(\cdot,\cdot)$, so

(29) $$t-\alpha=\pm a\int_0^z\frac{dz}{\sqrt{(a^2-z^2)(z^2-b^2)}}=F(\varphi,\ k),$$

where the amplitude $\varphi=am(t-\alpha,\ k)$, $m=k^2$, k is the modul of the elliptic function, m is the parameter of the elliptic function.

In the case under the consideration $a^2=1$, $b^2=\frac{W}{2}<1$ (in correspondance with the condition $-\pi<x<\pi$),

(30) $$k=\sqrt{\frac{W}{2}}\,,\quad m=\frac{W}{2}\,,\quad \sin\varphi=\frac{z}{\sqrt{\frac{W}{2}}}=\frac{\sin\frac{x}{2}}{k}$$

The solution of the equation (26) can be presented in the following form

(31) $$x = 2\arcsin\,[k\,\mathrm{sn}\,(t-\alpha)],$$

where sn(·) is sine of the amplitude (Jacobi's elliptic function).

Taking into account the dissipation, Eq. (26) becomes

(32) $$\frac{d^2x}{dt^2} + \sin x = -2\delta_d \frac{dx}{dt}.$$

Multiplying Eq. (32) with $\frac{dx}{dt}$ and integrating, we find

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \cos x\right] = -2\delta_d \frac{d}{dt}\int\left(\frac{dx}{dt}\right)^2 dt$$

or

(33) $$\frac{dW}{dt} = \frac{d}{dt}\left[-2\delta_d \int\left(\frac{dx}{dt}\right)^2 dt\right].$$

For a half of the period, from (30) and Eq. (33) we obtain

$$2\Delta m = \Delta W = -2\delta_d \int\left(\frac{dx}{dt}\right)^2 dt.$$

Using (31), we can write

(34) $$\frac{dx}{dt} = 2k\,\mathrm{cn}\,(t-\alpha)$$

and

$$\int\left(\frac{dx}{dt}\right)^2 dt = 4k^2 \int \mathrm{cn}^2\,(t-\alpha)dt,$$

where cn(·) is cosine of the amplitude (Jacobi's elliptic function).

Noting, that $\int \mathrm{cn}^2(t-\alpha)\,dt = \frac{1}{k^2}\,[\mathrm{E}\,(am\,(t-\alpha),\ k) - (1-k^2)(t-\alpha)]$ and $am\,[t-\alpha+2\mathrm{K}(k),\ k] = am(t-\alpha,\ k)+\pi,\ \mathrm{E}(\varphi+\pi, k) = \mathrm{E}(\varphi, k)+2E(k)$, where $\mathrm{E}(\cdot,\cdot)$ is in complete elliptic integral of second kind, hence $\int_0^{t+2K(k)} \mathrm{cn}^2(t-\alpha)\,dt = \frac{1}{k^2}$ $\times[2\mathrm{E}(k)-(1-k^2)2\mathrm{K}(k)]$.

For the half of period we have

(35a) $$2\Delta m = \Delta W = -16\delta_d\,[\mathrm{E}(k)-(1-k^2)\mathrm{K}(k)],$$

(35b) $$\Delta k = -\frac{4\delta_d}{k}\,[\mathrm{E}(k)-(1-k^2)\ \mathrm{K}(k)].$$

In the case of small k, $0<k\ll 1$, we can find

(36) $$\int_t^{t+2K(k)} \mathrm{cn}^2(t-\alpha)dt \simeq \int_0^{\pi} \cos^2(t-\alpha)\,dt = \frac{\pi}{2}.$$

Combining Eqs. (34), (35a) and (36), we obtain for the half of period

(37) $$\Delta m \simeq -2\pi\delta_d m, \quad \Delta k \simeq -\pi\delta_d k.$$

Let us introduce the following designations:

$$\Delta t_{4n} = t_{4n+1} - t_{4n}, \quad \Delta t_{4n+2} = t_{4n+3} - t_{4n+2}.$$

The bordering points are $x=\pm d$ and the semi-periods are symmetrical with respect of the time points $t_{4n,\max}$ and $t_{4n+2,\max}$ (see the Fig. 1).

For $\left\{\begin{array}{l} t=t_{4n,\max} \\ t=t_{4n+2,\max} \end{array}\right\}$ we have $\varphi=\left\{\begin{array}{c} + \\ - \end{array}\right\}\frac{\pi}{2}$.

Using (30) we can determine

(38) $$\varphi=\arcsin\left(\frac{\sin\frac{x}{2}}{k}\right).$$

Combining Eqs. (29) and (38) we find

(39) $$\Delta t_{4n}=2\left[F\left(\frac{\pi}{2},\,k\right)-F\left(\arcsin\frac{\sin\frac{d}{2}}{k},\,k\right)\right]$$

$$\simeq 2\left[K(k)-F\left(\frac{d}{2k},\,k\right)\right]\simeq 2\left[K(k)-\frac{d}{2k}\right],$$

(40) $$\Delta t_{4n+2}=2\left[F\left(\frac{\pi}{2},\,k\right)-F\left(\arcsin\frac{\sin\frac{d}{2}}{k},\,k\right)\right]=\Delta t_{4n}\simeq 2\left[K(k)-\frac{d}{2k}\right],$$

where $F(\cdot,\cdot)$ is incomplete elliptic integral of the first kind.

The expressions (39) and (40) are valid when $k>\sin\frac{d}{2}$.

Further we use the approach developed in [15] on the basis of stitching the solutions.

In the region $|x|<d$, noting that $d\ll 1$, we can use the linear approximation of the equation (26), i. e. $\frac{d^2x}{dt^2}+2\delta_d\frac{dx}{dt}+x\simeq\frac{P}{2d}\sin\nu t$ and its solution in the form

$$x=Re^{-\delta_d t}\sin[\omega(t-\gamma)]+\frac{\frac{P}{2d}}{\sqrt{(\nu^2-1)^2+(2\nu\delta_d)^2}}\sin(\nu t+\varphi_\nu),$$

vhere $\omega=\sqrt{1-\delta_d^2}$.

Let us assume that $\nu>1$, then $\varphi_\nu=\mathrm{arctg}\frac{2\nu\delta_d}{\nu^2-1}+\pi$.

When $0<\delta_d\ll 1$ and $\nu\gg 1$, the frequency $\omega\simeq 1$ and

$$x\simeq R\sin(t-\gamma)+\frac{\frac{P}{d}}{1-\nu^2}\sin\nu t,\quad \frac{dx}{dt}\simeq R\cos(t-\gamma)+\nu\frac{\frac{P}{2d}}{1-\nu^2}\cos\nu t.$$

Now, let us consider the region out of the acting zone $[-d,\,d]$, but closely to that zone, i. e. $|x|>d$, $|x|\simeq d$.

Under these conditions we can write:

$$x=2\arcsin[k\,\mathrm{sn}(t-\alpha)]=2\arcsin\{k\,\mathrm{sn}[2K(k)-(t-\alpha)]\}\simeq 2k[2K(k)-(t-\alpha)].$$

It follows that the moment t_{4n+1} can be found from the equation

(41) $$2k[2K(k)-(t_{4n+1}-\alpha)]\simeq d$$

when $\frac{dx}{dt} \simeq -2k$.

From the condition of lacking of x and $\frac{dx}{dt}$ interruption in the point $t = t_{4n+1}$, it follows

$$\begin{vmatrix} R\sin(t_{4n+1}-\gamma)+\frac{\frac{P}{2d}}{1-\nu^2}\sin \nu t_{4n+1} \simeq d, & \text{(42a)} \\ R\cos(t_{4n+1}-\gamma)+\nu\frac{\frac{P}{2d}}{1-\nu^2}\cos \nu t_{4n+1} \simeq -2k_{4n+1}. & \text{(42b)} \end{vmatrix}$$

Solving the system (42) we can obtain formulae for $R = R_{4n+1}$ and $\gamma = \gamma_{4n+1}$.
Analogically, when going out of the acting zone, i. e. for the point $t = t_{4n+2} = t_{4n+1} + \Delta t_{4n+1}$, where

(43) $$\Delta t_{4n+1} = t_{4n+2} - t_{4n+1},$$

we can write

$$\begin{vmatrix} R\sin(t_{4n+1}+\Delta t_{4n+1}-\gamma)+\frac{\frac{P}{2d}}{1-\nu^2}\sin[\nu(t_{4n+1}+\Delta t_{4n+1})] \simeq -d, & \text{(44a)} \\ R\cos(t_{4n+1}+\Delta t_{4n+1}-\gamma)+\gamma\frac{\frac{P}{2d}}{1-\nu^2}\cos[\nu(t_{4n+1}+\Delta t_{4n+1})] \simeq -2k_{4n+2}. & \text{(44b)} \end{vmatrix}$$

If

(45) $$\nu\Delta t_{4n+1} \ll 1,$$

the equations (42) give $\Delta t_{4n+1} \simeq \dfrac{2d}{-R\cos(t_{4n+1}-\gamma)+\nu\frac{\frac{P}{2d}}{1-\nu^2}\cos \nu t_{4n+1}}$.

Taking into account Eq. (42b), Expr. (43) becomes

(46) $$\Delta t_{4n+1} \simeq \frac{d}{k_{4n+1}}.$$

Let us introduce the designation

(47) $$\Delta k_{4n} \simeq k_{4n+1} - k_{4n}.$$

Comparison (47) with (37) reveals $\Delta k_{4n} \simeq -\pi\delta_d k_{4n}$.
Considering (39), we can write

(48) $$\Delta t_{4n} \simeq 2K(k_{4n}) - \frac{d}{k_{4n}}.$$

Using Eq. (44b), under the condition (45), we find

(49) $$k_{4n+2} \simeq \frac{1}{2}\Big\{-R_1\cos(t_{4n+1}-\gamma)+R\Delta t_{4n+1}\sin(t_{4n+1}-\gamma)$$

$$+\nu\frac{\frac{P}{2d}}{\nu^2-1}\cos\nu t_{4n+1}-\frac{\nu^2\frac{P}{2d}}{\nu^2-1}\Delta t_{4n+1}\sin\nu t_{4n+1}\Big\}.$$

Substituting Eqs. (42) and (46) in Eq. (49) we obtain

(50) $$k_{4n+2}\simeq k_{4n+1}-\frac{P}{4k_{4n+1}}\sin\nu t_{4n+1}.$$

Analogically we can write the following equations

(51) $$\Delta t_{4n+2}\simeq 2K(k_{4n+2})-\frac{d}{k_{4n+2}},$$

(52) $$\Delta k_{4n+2}\simeq-\pi\delta_d k_{4n+2},$$

(53) $$k_{4n+3}=k_{4n+2}+\Delta k_{4n+2},$$

$$t_{4n+3}=t_{4n+2}+\Delta t_{4n+2}.$$

For the region $4n+3\rightarrow 4n+4$ (see Fig. 1) we have ($R=R_{4n+3}$, $\gamma=\gamma_{4n+3}$)

(54a) $$R\sin(t_{4n+3}-\gamma)-\frac{\frac{P}{2d}}{\nu^2-1}\sin\nu t_{4n+3}\simeq -d,$$

(54b) $$R\cos(t_{4n+3}-\gamma)-\nu\frac{\frac{P}{2d}}{\nu^2-1}\cos\nu t_{4n+3}\simeq 2k_{4n+3},$$

$$t_{4n+4}=t_{4n+3}+\Delta t_{4n+3},$$

(55a) $$R\sin(t_{4n+3}+\Delta t_{4n+3}-\gamma)-\frac{\frac{P}{2d}}{\nu^2-1}\sin\nu(t_{4n+3}+\Delta t_{4n+3})\simeq d,$$

(55b) $$R\cos(t_{4n+3}+\Delta t_{4n+3}-\gamma)-\nu\frac{\frac{P}{2d}}{\nu^2-1}\cos\nu(t_{4n+3}+\Delta t_{4n+3})\simeq 2k_{4n+4}.$$

Assuming, that

(56) $$\nu\Delta t_{4n+3}\ll 1,$$

and combining Eqs. (54a) and (55a) we find

$$\Delta t_{4n+3}\simeq\frac{2d}{R\cos(t_{4n+3}-\gamma)-\nu\frac{\frac{P}{2d}}{\nu^2-1}\cos\nu t_{4n+3}}.$$

From (54b) it follows

(57) $$\Delta t_{4n+3}\simeq\frac{d}{k_{4n+3}},\quad t_{4n+4}=t_{4n+3}+\Delta t_{4n+3}.$$

Taking into account Eqs. (52) and (50), we can write

$$k_{4n+2}\simeq k_{4n}-\pi\delta_d k_{4n}-\frac{P}{4k_{4n}}\sin\nu t_{4n+2}.$$

Considering the condition (56), Eq. (55b) can be rewritten

(58) $$k_{4n+4} \simeq \frac{1}{2} \left\{ R\cos(t_{4n+3}-\gamma_{4n+3}) - \nu \frac{\frac{P}{2d}}{\nu^2-1} \cos \nu t_{4n+3} \right.$$

$$\left. - R\Delta t_{4n+3} \sin(t_{4n+3}-\gamma) + \nu^2 \frac{\frac{P}{2d}}{\nu^2-1} \Delta t_{4n+3} \sin \nu t_{4n+3} \right\}.$$

Using Eqs. (54) and (58) we can write

(59) $$k_{4n+4} \simeq k_{4n+3} + \frac{P}{4k_{4n+3}} \sin \nu t_{4n+3}.$$

Comparison (59) with (52) and (53) reveals

$$k_{4n+4} \simeq k_{4n+2} - \pi\delta_d k_{4n+2} + \frac{P}{4k_{4n+2}} \sin \nu t_{4n+4}.$$

Combining Eqs. (46) and (48) we find

$$t_{4n+2} \simeq \left[2K(k_{4n}) - \frac{d}{k_{4n}} \right] + \frac{d}{k_{4n+1}} + t_{4n} \simeq 2K(k_{4n}) + t_{4n}.$$

In analogue from Eqs (51) and (57) we can obtain

$$t_{4n+4} = \left[2K(k_{4n+2}) - \frac{d}{k_{4n+2}} \right] + \frac{d}{k_{4n+3}} + t_{4n+2} \simeq 2K(k_{4n+2}) + t_{4n+2}.$$

In the long run we have obtained the following system of equations:

(60a) $$t_{4n+2} \simeq t_{4n} + 2K(k_{4n}),$$

(60b) $$k_{4n+2} \simeq k_{4n} - \pi\delta_d k_{4n} - \frac{P}{4k_{4n}} \sin \nu t_{4n+2},$$

(60c) $$t_{4n+4} \simeq t_{4n+2} + 2K(k_{4n+2}),$$

(60d) $$k_{4n+4} \simeq k_{4n+2} - \pi\delta_d k_{4n+2} + \frac{P}{4k_{4n+2}} \sin \nu t_{4n+4}.$$

The spectrum of possible stationary amplitudes of continuous oscillations is determined by the expression:

(61) $$\nu(t_{4n+4} - t_{4n}) = 2\pi N,$$

where $N = 1, 2, 3, \ldots$ is the ratio of frequency division.

The equation (61) can be written also in the form

$$\nu[K(k_{2n}) + K(k_{2n+2})] = \pi N.$$

Below we show that N has to be an odd number.

Designating five successive time points as t_0, t_1, t_2, t_3 and t_4 and corresponding values of k as k_0, k_1, k_2, k_3, and k_4 (in analogue to as it has been done in Fig. 1), we can write the following conditions for the stationary mode

(62a) $$k_4 = k_0,$$

(62b) $$\nu(t_4 - t_0) = 2\pi N.$$

The equation (62b) follows from the condition of oscillation synchronization with the external excitment.

The Eqs. (60) can be rewritten

$$k_2=k_0-\pi\delta_d k_0-\frac{P}{4k_0}\sin\nu t_2,\quad k_4=k_2-\pi\delta_d k_2+\frac{P}{4k_0}\sin\nu t_0,$$

$$t_2=t_0+2\mathrm{K}(k_0),\quad t_4=t_2+2\mathrm{K}(k_2).$$

If we consider the condition of symmetry between the upper $\{4n\to 4n+2\}$ and lower $\{4n+2\to 4n+4\}$ periods, we can find that it is possible to have the symmetry only if N is an odd number, i. e. $N=2l+1$, $l=0, 1, 2, 3, \ldots$, and if the next equality is fulfilled: $\sin\nu[t+2\mathrm{K}(k)]=-\sin\nu t$.

From Eqs. (41) and (42) it follows $2\nu\mathrm{K}(k_0)=2\pi\left(l+\frac{1}{2}\right)$ and

(63a) $$\sin\nu t_0=-\sin\nu t_2,$$

(63b) $$\cos\nu t_0=-\cos\nu t_2.$$

Combining Eqs. (35b), (62a) and (63) we can write

(64) $$-\frac{16\delta_d}{P}\cdot[\mathrm{E}(k_0)-(1-k_0^2)\mathrm{K}(k_0)]-\sin\nu t_0=0,$$

which in the case of $k\to 0$ becomes

(65) $$\frac{4\pi\delta_d k_0^2}{P}-\sin\nu t_0=0.$$

Solving Eq. (65) we can determine the initial phase t_0.

Apparently there exists an excitation threshold,

(66) $$|P|>4\pi\delta_d k^2.$$

For the P values above the value $P=4\pi\delta_d k^2$, a disctetization of the possible stationary oscillating amplitudes appears.

It is interesting to note that the threshold condition (66) coincides with the analogical condition (21), obtained for the case when the nonlinearity of the external harmonic excitement is presented by δ-function.

As we have assumed the solution symmetry, for the sake of system stability examination it is enough to consider only a half or the period.

From Eq. (60b) we determine the variation

$$\delta k_{4n+2}=\left(1-\pi\delta_d+\frac{P}{4k_{4n}^2}\sin\nu t_{4n+2}\right)\delta k_{4n}-\nu\frac{P}{4k_{4n}}(\cos\nu t_{4n+2})\delta t_{4n+2}.$$

Taking into account the relation [16] $\frac{d\mathrm{K}(k)}{dk}=\frac{1}{k}\left[\frac{\mathrm{E}(k)}{1-k^2}-\mathrm{K}(k)\right]$ and also the approximate formulae $\mathrm{E}(k)=\frac{\pi}{2}\left(1-\frac{k^2}{4}\right)+O(k^4)$, $\mathrm{K}(k)=\frac{\pi}{2}\left(1+\frac{k^2}{4}\right)+O(k^4)$ we can write

(67) $$\frac{d\mathrm{K}(k)}{dk}=\frac{\pi}{4}k+O(k^3).$$

Considering (67), from (60c) we can find the following variation:

$$\delta t_{4n+4}=\delta t_{4n+2}+2\frac{d\mathrm{K}(k)}{dk}\delta k_{4n+2}=\frac{\pi}{2}k_{4n+2}\left(1-\pi\delta_d+\frac{P}{4k_{4n}^2}\sin\nu t_{4n+2}\right)\delta k_{4n}$$

$$+\left(1-\frac{\pi}{8}\nu P\frac{k_{4n+2}}{k_{4n}}\cos\nu t_{4n+2}\right)\delta t_{4n+2}.$$

Remembering that $k_{4n+2}=k_{4n}=k_0$, $\sin\nu t_{4n+2}=-\sin\nu t_0$, $\cos\nu t_{4n+2}=-\cos\nu t_0$, from Eq. (64) follows that $\frac{P\sin\nu t_{4n+2}}{4k_{4n}^2}=-\frac{P\sin\nu t_0}{4k_0^2}=-\pi\delta_d$, so, we can write

$$\begin{cases}\delta k_{4n+2}=(1-2\pi\delta_d)\delta k_{4n}+\nu\frac{P}{4k_0}(\cos\nu t_0)\delta t_{4n+2},\\ \delta t_{4n+4}=\frac{\pi}{2}k_0(1-2\pi\delta_d)\delta k_{4n}+\left(1+\frac{\pi}{8}\nu P\cos\nu t_0\right)\delta t_{4n+2}.\end{cases}$$

Let us assume that $\delta k_{4n+2}=\lambda\delta k_{4n}$ and $\delta t_{4n+4}=\lambda\delta t_{4n+2}$.
Hence we can write

$$\begin{cases}(1-2\pi\delta_d-\lambda)\,\delta k_{4n}+\nu\frac{P}{4k_0}(\cos\nu t_0)\delta t_{4n+2}=0,\\ \frac{\pi}{2}k_0(1-2\pi\delta_d)\delta k_{4n}+\left(1+\frac{\pi}{8}\nu P\cos\nu t_0-\lambda\right)\delta t_{4n+2}=0.\end{cases}$$

The characteristic equation has the form

$$\lambda^2-\lambda(2-2\pi\delta_d+\frac{\pi}{8}\nu P\cos\nu t_0)+(1-2\pi\delta_d)=0$$

and its solution is

$$\lambda_{1,2}=1-\pi\delta_d+\frac{\pi}{16}\nu P\cos\nu t_0\pm\sqrt{\left(1-\pi\delta_d+\frac{\pi}{16}\nu P\cos\nu t_0\right)^2-1+2\pi\delta_d}.$$

The stability condition is: $|\lambda_{1,2}|<1$.

Apparently, the solution is stable when the following condition is satisfied: $P\nu\cos\nu t_0<0$.

Generally, we have proved that in the system under consideration oscillations with an amplitude from a possible set of stable amplitudes can be excited.

The spectrum of the symmetrical solution amplitudes can be expressed as $2\nu K(k_0)=2\pi\left(l+\frac{1}{2}\right)$, $l=0, 1, 2, 3, \ldots$, which gives the spectrum of amplitudes k_0, $K(k_0)=\left(l+\frac{1}{2}\right)\frac{\pi}{\nu}$, $l=0, 1, 2, 3, \ldots$ and an odd ratio of frequency division $N=2l+1$, $l=0, 1, 2, 3, \ldots$

Conclusion

It should be noted that the relation $\nu=N\omega_0$ is complied with in all cyclic accelerators; there ν is the accelerating high-frequency field frequency, ω_0 is charged particles rotating frequency, and N is acceleration multiplicity reaching tens and hundreds. That is why, the above discussed stationary oscillations are analogous to the movement of "equilibrium" particle in cycloaccelerator. Particles, close to the equilibrium, in cycloaccelerators perform slow phase oscillations. Their analogue in our system is the fluctuating approximation to stationary values of the oscillation amplitude and phase. In our problem, the phase oscillations damping is determined by the friction coefficient δ_d, while in charged particles accelerators damping is result of radioemission. Here, ν, N and ω_0 are constants, however in the cycloaccelerators the process of charged particles acceleration is accompanied by increase of ν (phasotron), N (microtron), ω_0 (synchrotron) or ν and ω_0 (synchrophasotron). Injection in acceleration mode (for accelerators) and in stationary oscillations mode (our system) represents a separate problem [14].

The presented mechanism of continuous oscillations excitation allow to examine from this position the processes of plasma particle interaction with electromagnetic waves. For example, equation of (1) form is obtained with right-hand $F_0(x, t_r) = E \cos k_N x \sin \nu t_r$; in this case $\delta_d = \omega_i$ is ion-electron-neutral atoms collision frequency and $\omega_0 = \frac{eB}{MC}$ is ion cyclotron frequency, e is electron charge, M is ion mass, C is light velocity. This is the case of electromagnetic wave interaction with particle in cylindric waveguide with longitudinal magnetic field B and E type wave. If, for example, $\nu = N\omega_0$ UHF oscillation is transformed into low-frequency oscillation ω_0, then the corresponding correlations between E and E_0 (longitudinal electric field) is: $E = \frac{e}{M} E_0 \frac{\Omega_p}{\nu} = \frac{e}{M} E_0 \frac{C}{V_A}$, where $\Omega_p = 4\pi C^2 \frac{n_N}{M}$ is Langmuir plasma frequency, $V_A = \frac{B}{\sqrt{4\pi N M}}$ is Alfvén velocity, n_N is plasma density. The condition for plasma heating is defined as $E_0 > \frac{M}{l_R} \cdot \frac{\omega_i \omega_0^2 R_0 N}{\omega_p}$, where R_0 and l_R are the waveguide radius and lenght, $N = \frac{\nu}{\omega_0}$, $\omega_p = 4\pi l_R^2 \frac{n_N}{M}$.

Examination of the process of energy transformation efficiency in the centimeter, IR and optical wavebands in low-frequency oscillations demonstrates the potentials for generation of powerful low-frequency waves in the Solar system near-planet space.

So, simple modelling systems and mechanisms of oscillation excitation are presented that may contribute to the revealing of mechanisms of planetary magnetosphere radiosources generation and wave interactions mechanisms in the Earth ionosphere and magnetosphere as well as the excitation of VLF waves in the near-Earth space.

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Возбуждение „Квантованных" колебаний под воздействием внешней неоднородной силы

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(Резюме)

Аналитически представлено явление возбуждения незатухающих колебаний с амплитудой, принадлежащей к дискретному ряду возможных устойчивых амплитуд для двух случаев — во-первых, когда внешнее воздействие, представленное δ-функцией, прикладывается к нижней равновесной точке траектории колебаний и, во-вторых — когда внешняя гармоническая сила воздействует в заданной зоне траектории с конечной протяженностью.

Представленные модельная система и механизм возбуждения колебаний могут найти применение в работе по выявлению механизмов генерации радиоисточников в магнитосферах планет и механизмов взаимодействия волн в йоносфере и магнитосфере Земли, а также возбуждения НЧ волн в околоземном пространстве.